Basic Math
A Self-Tutorial

by

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Lesson 4:
Properties Of Numbers

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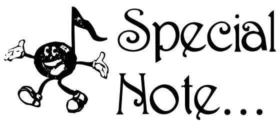
INTRODUCTION

All real numbers, variables, and algebraic expressions follow certain properties. They can range from "the obvious" to a little complicated. This lesson will list them and provide some examples where appropriate. Most examples will be with numbers and simple variables (I want to save the major algebraic manipulations for when I am formally doing Algebra Lessons).



When explaining these properties, I will let ***a***, ***b***, ***c*** represent real numbers, variables or algebraic expressions. For simplicity, I will usually just say "number" or "value."

PROPERTIES OF EQUALITY

✦ The ***Reflexive Property of Equality*** (also called the Identity Property) states that a number is equal to … *<insert drumroll here>* ITSELF! Symbolically:

$$a = a$$

✦ The ***Symmetric Property of Equality*** states that if one value is equal to another, then that second value is the same as the first. Symbolically:

$$\text{If } a = b, \text{ then } b = a$$

This is a handy rule to know when solving for a variable. Often, you may get, for example, an answer like:

$$3 = x$$

By the Symmetric Property, you can "switch" the places of the values:

$$x = 3$$

Which is a much "nicer looking" answer. It is traditional to have the variable on the left side of the equation for the "final answer."

✦ The ***Transitive Property of Equality***: If one value is equal to a second, and the second happens to be the same as a third value, then we can conclude the first value must also equal the third. Symbolically:

If $a = b$ and $b = c$
then $a = c$

✦ The ***Substitution Property*** (also called the Substitution Principle) states: if one value is equal to another, then the second value can be used in place of the first in any algebraic expression dealing with the first value. Symbolically:

If $a = b$, then b can be substituted for a in any expression dealing with a.

Of course, we could switch things around and have said:

If $a = b$, then a can be substituted for b in any expression dealing with b.

So this Property works both ways.

✦ The ***Additive Property of Equality***: We can add equal values to both sides of an equation without changing the validity of the equation.

If $a = b$,
then $a + \boldsymbol{c} = b + \boldsymbol{c}$
and $\boldsymbol{c} + a = \boldsymbol{c} + b$

The reverse of this is called:

✦ The ***Cancellation Law of Addition***:

If $a + c = b + c$
then $a = b$

In other words, we can subtract c from both sides of an equation, to "cancel" it, leaving only a and b.

This also works for multiplication:

✦ The ***Multiplicative Property of Equality***: We can multiply equal values to both sides of an equation without changing the validity of the equation.

If	$a = b,$
then	$a\mathbf{c} = b\mathbf{c}$
and	$\mathbf{c}a = \mathbf{c}b$

The reverse of this is called:

✦ The ***Cancellation Law of Multiplication***:

If	$ac = bc$ and $c \neq 0$ <c is not equal to zero>
then	$a = b$

In other words, so long as c is not zero, we can divide both sides of an equation by c to "cancel" it, leaving only a and b.

✦ The ***Zero-Factor Property*** (also called the Zero-Factor Principle or Zero-Product Theorem) states: if two values that are being multiplied together equal zero, then one of the values, or both of them, must equal zero. Symbolically:

If	$ab = 0$
then	$a = 0$, $b = 0$, or both a and $b = 0$

This property may be stated in the "reverse" form:

If	$a = 0$ or $b = 0$
then	$ab = 0$

Properties of Inequality

✦ The ***Law of Trichotomy*** states that for any two values, only ONE of the following can be true about these values:

① They are equal.

② The first has a smaller value than the second.

③ The first has a larger value than the second.

Symbolically:

Given any numbers a and b:

$a = b$ or $a < b$ or $a > b$

✦ The ***Transitive Property of Inequality***: If one value is smaller than a second, and the second is less than a third, then we can conclude the first value is smaller than the third. Symbolically:

If $a < b$ and $b < c$
then $a < c$

There are a few other properties of inequalities, but these will be covered in the Algebra Lesson: “Solving Linear Inequalities”

PROPERTIES OF ABSOLUTE VALUE

✷ Rule ①: $|a| \geqslant 0$ <*All absolute values are non-negative (zero or positive)*>

✷ Rule ②: $|-a| = |a|$ <*The absolute value of the opposite of a number is the same as the absolute value of the number*>

✷ Rule ③: $|ab| = |a||b|$ <*You can multiply numbers "inside" the absolute value* <u>*or*</u> *take the absolute value of the individual numbers first,* <u>*then*</u> *multiply the numbers*>

✷ Rule ④: $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, $b \neq 0$ <*So long as b is not equal to zero, you can divide numbers inside the absolute value bars* <u>*or*</u> *take the absolute value of the individual numbers first,* <u>*then*</u> *divide*>

PROPERTIES OF NUMBERS

✦ *CLOSURE* ✦

✦ The ***Closure Property of Addition*** states that when you add real numbers to other real numbers, the sum is also real. Addition is a "closed" operation. Symbolically:

$$a + b = \text{a real number}$$

✦ The ***Closure Property of Multiplication*** states that when you multiply real numbers to other real numbers, the product is… you guessed it… a REAL number! Multiplication is a "closed" operation. Symbolically:

$$a \cdot b = \text{a real number}$$

The real numbers are closed with respect to addition and multiplication. This means that when you are adding or multiplying real numbers, the result you will get will also be a real number.

What would NOT be closed? An example of this would be subtraction of natural numbers. To illustrate:

3 is a natural number. So is 5. But when we subtract these numbers, we get:

$$3 - 5 = -2$$

–2 is NOT a natural number, so subtraction is not a closed operation for natural numbers.

✦ *COMMUTATIVITY* ✦

✦ The ***Commutative Property of Addition*** states that it does not matter the order in which numbers are added together. For example: 2 + 3 is the same as 3 + 2. Symbolically:

$$a + b = b + a$$

✦ The ***Commutative Property of Multiplication*** states that it does not matter the order in which numbers are multiplied together. For example: 2 × 3 is the same as 3 × 2. Symbolically:

$$a \cdot b = b \cdot a$$
or
$$ab = ba$$

☞ *By The Way...* The Commutative Property doe NOT apply to subtraction or division. For example 5 – 2 $\neq$ 2 – 5 and $\frac{2}{1} \neq \frac{1}{2}$.

✦ *ASSOCIATIVITY* ✦

✦ The ***Associative Property of Addition*** states that when we wish to add three <or more!> numbers, it does not matter how we group them together for adding purposes. The parentheses can be placed as we wish. For example: (1 + 2) + 3 will give the exact same result as 1 + (2 + 3). We can "associate" <group> the first two numbers together, then add the third; or, if we wish, we can "associate" the last two values, add them up, *then* add the first value to this result. In any case, the answers are equal. Symbolically:

$$\left(a+b\right)+c=a+\left(b+c\right)$$

✦ The ***Associative Property of Multiplication*** states that when we wish to multiply three <or more!> numbers, it does not matter how we group them together for multiplication purposes. The parentheses can be placed as we wish. For example: (2 × 3) × 4 will give the exact same result as 2 × (3 × 4). We can "associate" the first two numbers together, then multiply the third; or, if we wish, we can "associate" the last two values, multiply them up, *then* multiply the first value to this result. In any case, the answers are equal. Symbolically:

$$\left(a \cdot b\right) \cdot c = a \cdot \left(b \cdot c\right)$$
or
$$\left(ab\right)c = a\left(bc\right)$$

☞ *By The Way...* The Associative Property does NOT apply to subtraction or division.

✦ *IDENTITY* ✦

✦ The ***Identity Property of Addition*** (also called the Additive Identity Property) states there exists a special number, called the "additive identity," when added to any other number, then that other number will still "keep its identity" and remain the same. Zero is this unique additive identity. Symbolically:

$$a + 0 = a$$

This is also true if the summands were to switch places:

$$0 + a = a$$

✦ The ***Identity Property of Multiplication*** (also called the Multiplicative Identity Property) states there exists a special number, called the "multiplicative identity," when multiplied to any other number, then that other number will still "keep its identity" and remain the same. The number 1 is this unique multiplicative identity. Symbolically:

$$a \cdot 1 = a$$
and
$$1 \cdot a = a$$

✦ *INVERSE* ✦

✦ The ***Inverse Property of Addition*** (also called the Additive Inverse Property) states that for every real number, there exists another real number that is called its opposite, such that, when added together, you get the additive identity. In other words, when you add opposites, they "cancel" out leaving only zero. Symbolically:

$$a + (-a) = 0$$
and
$$(-a) + a = 0$$

♪ **Note:** $-a$ represents the opposite, or negation of the number a.

♪ **Another Note:** If a is a number, its additive inverse is: $-a$.

✦ The ***Inverse Property of Multiplication*** (also called the Multiplicative Inverse Property) states that for every real number, except zero, there is another real number that is called its multiplicative inverse, or reciprocal, such that, when multiplied together, you get the multiplicative identity (the number 1). In other words, when you multiply reciprocals, they "cancel" out to get 1. Symbolically:

$$a \cdot \frac{1}{a} = 1$$

and

$$\frac{1}{a} \cdot a = 1$$

♪ **Note:** $\frac{1}{a}$ represents the reciprocal of a.

♪ **Another Note:** If "a" is a number, its multiplicative inverse is: $\frac{1}{a}$.

☞ *By The Way...* The number 0 is the ONLY real number that does *NOT* have a multiplicative inverse, since $\frac{1}{0}$ is undefined.

✦ *DISTRIBUTIVITY* ✦

<Try saying that 3 times fast!>

✦ The ***Distributive Law of Multiplication Over Addition*** (or simply: "The Distributive Law" or "Distributive Property") states that multiplying a number by a sum of numbers is the same as multiplying each number in the sum individually, then adding up our products.
For Example:

$$\left.\begin{aligned} &5(7+3) \\ &=5(10) \\ &=50 \end{aligned}\right\} \text{ is the same as } \left\{\begin{aligned} &5(7)+5(3) \\ &=35+15 \\ &=50 \end{aligned}\right.$$

Symbolically:

$$\begin{aligned} a(b+c) &= a \cdot b + a \cdot c \\ &= ab + ac \end{aligned}$$

A good way to remember this property is to imagine the *a* being "distributed" to each item within the parentheses:

First: $a\,(\,b + c\,)$ Then: $a\,(\,b + c\,)$

The *a* will multiply with the *b* and *c* to get $a \cdot b$ and $a \cdot c$. Add these values up to get the final answer of $ab + ac$.

Some textbooks "split" this Law up into two parts:

✷ ***Left Distributive Property***: <This is just the one we've been doing>

$$a\left(b+c\right)=ab+ac$$

✷ ***Right Distributive Property***:

$$\left(a+b\right)c=ac+bc$$

The *c* "distributes" itself to the values inside the parentheses.

The Distributive Law also works with subtraction:

✷ The ***Distributive Law of Multiplication Over Subtraction***:

$$a\left(b-c\right)=ab-ac$$

The Distributive Law will work for more than two terms within the parentheses. For example:

Given: 2(1 + 3 + 5 + 7), we can "distribute" the 2:

$$\begin{aligned} 2(1+3+5+7) &= 2(1)+2(3)+2(5)+2(7) \\ &= 2+6+10+14 \\ &= 32 \end{aligned}$$

In general, we call this:

✷ The ***General Distributive Property***:

$$a\left(b_1 + b_2 + b_3 + \ldots + b_n\right) = ab_1 + ab_2 + ab_3 + \ldots + ab_n$$

b_1 is the first term within the parentheses, b_2 the second, b_3 the third, and so on, until you reach the "n^{th}" term, b_n, which is the last one.

✷ The ***Negation Distributive Property***: (also called the Opposite of a Sum Property) If you negate (or find the opposite) of a sum, just "change the signs" of whatever is inside the parentheses. Symbolically:

$$-\left(a+b\right) = \left(-a\right) + \left(-b\right) = -a - b$$

Here is a summary of our properties:

PROPERTIES OF NUMBERS

If *a*, *b* and *c* are real numbers (or variables or algebraic expressions), then all of the following properties hold:

✷ The ***Reflexive Property of Equality***: $a = a$

✷ The ***Symmetric Property of Equality***: If $a = b$, then $b = a$

✷ The ***Transitive Property of Equality***: If $a = b$ and $b = c$, then $a = c$

✷ The ***Substitution Property***: If $a = b$, then *b* can be substituted for *a* in any expression dealing with *a* ***and*** *a* can be substituted for *b* in any expression dealing with *b*.

✷ The ***Additive Property of Equality***: If $a = b$, then $a + c = b + c$
and $c + a = c + b$

✷ *The* ***Cancellation Law of Addition***: *If* $a + c = b + c$, *then* $a = b$

✷ The ***Multiplicative Property of Equality***: If $a = b$, then $ac = bc$
and $ca = cb$

✷ ***Cancellation Law of Multiplication***: If $ac = bc$ and $c \neq 0$, then $a = b$

✷ The ***Zero-Factor Property***: If $ab = 0$, then $a = 0$ or $b = 0$ (or both are 0)

✷ The ***Law of Trichotomy:*** For any two real numbers a and b, only one of the following can be true: $a = b$, $a < b$, or $a > b$.

✷ The ***Transitive Property of Inequality***: If $a < b$ and $b < c$, then $a < c$

✷ ***Properties of Absolute Value***:

$$|a| \geqslant 0 \qquad |-a| = |a| \qquad |ab| = |a||b| \qquad \left|\frac{a}{b}\right| = \frac{|a|}{|b|},\ b \neq 0$$

✷ The ***Closure Property of Addition***: $a + b$ is a real number.

✷ The ***Closure Property of Multiplication***: ab is a real number.

✷ The ***Commutative Property of Addition***: $a + b = b + a$

✷ The ***Commutative Property of Multiplication***: $ab = ba$

✷ The ***Associative Property of Addition***: $(a+b)+c = a+(b+c)$

✷ The ***Associative Property of Multiplication***: $(ab)c = a(bc)$

✷ The ***Identity Property of Addition***: $a + 0 = 0 + a = a$

0 is the Additive Identity.

✷ The ***Identity Property of Multiplication***: $a \cdot 1 = 1 \cdot a = a$

1 is the Multiplicative Identity.

✷ The ***Inverse Property of Addition***: $a + (-a) = (-a) + a = 0$

The Additive Inverse of a number a is $-a$.

✷ The ***Inverse Property of Multiplication***: $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$

The Multiplicative Inverse of a number a is $\frac{1}{a}$.

✷ The ***Distributive Law of Multiplication Over Addition***:

$$a\left(b+c\right)=ab+ac$$

or

$$\left(a+b\right)c=ac+bc$$

✷ The ***Distributive Law of Multiplication Over Subtraction***:

$$a\left(b-c\right)=ab-ac$$

✷ The ***General Distributive Property***:

$$a\left(b_1+b_2+b_3+\ldots+b_n\right)=ab_1+ab_2+ab_3+\ldots+ab_n$$

✷ The ***Negation Distributive Property***: $-\left(a+b\right)=\left(-a\right)+\left(-b\right)=-a-b$

LESSON 4 QUIZ

① Find the additive inverses of the following:

- ✷ –5: ____
- ✷ $\frac{2}{3}$: ____
- ✷ –1: ____
- ✷ 0: ____

② Find the multiplicative inverses of the following:

- ✷ –5: ____
- ✷ $\frac{2}{3}$: ____
- ✷ –1: ____
- ✷ 0: ____

③ What is the additive identity?

④ What is the multiplicative identity?

⑤ Do all numbers have an additive inverse?

⑥ Do all numbers have a multiplicative inverse?

⑦ Complete each equation by filling in the box with what is necessary to perform the property that is asked for.

✷ $-u + u = \square$	Additive Inverse
✷ $8 \times 7 = \square$	Commutative Property of Multiplication

* $5(w-y)=$ ______ Distributive Law
* $-3+(6+2)=$ ______ Associative Property of Addition
* $z=$ ______ Reflexive Property of Equality
* $a \nless b$, $a \neq b$, therefore: ______ Trichotomy Property

⑧ Write in the blank the property that is being illustrated.

* $m\left(\frac{1}{m}\right)=1$ ______________________________
* Since $\sqrt{3}$ and e are real numbers, so is $\sqrt{3}+e$ ______________________________
* $2+x^2=x^2+2$ ______________________________
* $(z+7)+2=z+(7+2)$ ______________________________
* $(y)(1)=y$ ______________________________
* If $x = y$ and $y = 5$, then $x = 5$ ______________________________
* $\sqrt{2}+0=\sqrt{2}$ ______________________________
* $-(x+2)=-x-2$ ______________________________

⑨ Same as above, but these are tougher or sneakier ( ). There may be more than one answer.

* $\sqrt{3}(2+x)=\sqrt{3}(x+2)$ ______________________________
* $(ab)c=(ba)c$ ______________________________
* $[2+(x-1)]y=2y+(x-1)y$ ______________________________

✷ $\left(\frac{1}{x^2+4}\right)\left(x^2+4\right)=1$ ______________________________

✷ $\left(x+y\right)+z=z+\left(x+y\right)$ ______________________________

✷ $\left(1\right)\left(1\right)=1$ ______________________________

✷ $5+w+\left(-w\right)=5$ ______________________________

✷ $\left(2a\right)\left(bc\right)=2\left(ab\right)c$ ______________________________

✷ $\left|\frac{-2}{3}\right|=\frac{\left|-2\right|}{\left|3\right|}=\frac{2}{3}$ ______________________________

✷ $\left(x+1\right)\left(y+2\right)=\left(x+1\right)\left(y\right)+\left(x+1\right)\left(2\right)$ ______________________________

✷ $1\cdot\left(y-2\right)=y-2$ ______________________________

⑩ Use the commutative properties to rewrite each of the following expressions:

✷ $x + 5 =$ __________ ✷ $pq =$ ________

✷ $2y + 8 =$ ___________ ✷ $2 - ab =$ __________

⑪ Use the associative properties to rewrite each of the following expressions:

✷ $3+\left(w+z\right) =$ __________ ✷ $3\left(wz\right) =$ __________

⑫ Use the distributive property to rewrite and simplify, if possible, each of the following expressions:

✷ $-2\left(x+3\right) =$ ____________ ✷ $-\left(2y-9\right) =$ ____________

Answers On Next Page…

Answers

① Find the additive inverses of the following:

- ✷ -5: 5
- ✷ $\frac{2}{3}$: $-\frac{2}{3}$
- ✷ -1: 1
- ✷ 0: 0

② Find the multiplicative inverses of the following:

- ✷ -5: $-\frac{1}{5}$
- ✷ $\frac{2}{3}$: $\frac{3}{2}$
- ✷ -1: -1
- ✷ 0: *None*

③ What is the additive identity? 0

④ What is the multiplicative identity? 1

⑤ Do all numbers have an additive inverse? Yes.

⑥ Do all numbers have a multiplicative inverse? No, zero does not.

⑦ Here are the Completed Equations (boxes removed for better visibility):

✷ $-u + u = 0$	Additive Inverse
✷ $8 \times 7 = 7 \times 8$	Commutative Property of Multiplication

✱ $5(w-y) =$ $5w - 5y$ Distributive Law

✱ $-3+(6+2) =$ $(-3 + 6) + 2$ Associative Property of Addition

✱ $z = z$ Reflexive Property of Equality

✱ $a \not< b$, $a \neq b$, therefore: $a > b$ Trichotomy Property

⑧ Write in the blank the property that is being illustrated.

✱ $m\left(\frac{1}{m}\right)=1$ Inverse Property of Mutiplication

✱ Since $\sqrt{3}$ and e are real numbers,
so is $\sqrt{3}+e$ Closure Property of Addition

✱ $2+x^2 = x^2+2$ Commutative Property of Addition

✱ $(z+7)+2 = z+(7+2)$ Associative Property of Addition

✱ $(y)(1) = y$ Identity Property of Multiplication

✱ If $x = y$ and $y = 5$, then $x = 5$ Transitive Property of Equality

✱ $\sqrt{2}+0=\sqrt{2}$ Identity Property of Addition

✱ $-(x+2) = -x-2$ Negation Distributive Property

⑨ Same as above, but these are tougher or sneakier ( ). There may be more than one answer.

✱ $\sqrt{3}(2+x) = \sqrt{3}(x+2)$ Commutative Property of Addition

✱ $(ab)c = (ba)c$ Commutative Property of Multiplication

✱ $[2+(x-1)]y = 2y+(x-1)y$ Distributive Law (or Right Distributive Property)

✷ $\left(\frac{1}{x^2+4}\right)\left(x^2+4\right)=1$ Inverse Property of Multiplication

✷ $(x+y)+z=z+(x+y)$ Commutative Property of Addition (Many mistakenly think this is the Associative Property, but it isn't.)

✷ $(1)(1)=1$ This is the Identity Property of Multiplication, but also the Inverse Property of Multiplication since 1 is its own reciprocal.

✷ $5+w+(-w)=5$ Inverse Property of Addition (with the *w*'s)

✷ $(2a)(bc)=2(ab)c$ Associative Property of Multiplication

✷ $\left|\frac{-2}{3}\right|=\frac{|-2|}{|3|}=\frac{2}{3}$ Property of Absolute Value

✷ $(x+1)(y+2)=(x+1)(y)+(x+1)(2)$ Distributive Law [the $(x+1)$ is being distributed to the *y* and the 2]

✷ $1\cdot(y-2)=y-2$ Identity Property of Multiplication (but it's also the Distributive Law of Multiplication over Subtraction, since we can intrepret this to mean the 1 is being distributed to the *y* and the 2)

⑩ Use the commutative properties to rewrite each of the following expressions:

✷ $x + 5 = 5 + x$

✷ $pq = qp$

✷ $2y + 8 = 8 + 2y$ or $y(2) + 8$ or $8 + y(2)$

✷ $2 - ab = 2 - ba$

⑪ Use the associative properties to rewrite each of the following expressions:

* $3+(w+z)$ = $(3 + w) + z$
* $3(wz)$ = $(3w)z$

⑫ Use the distributive property to rewrite and simplify, if possible, each of the following expressions:

* $-2(x+3)$ = $-2x + (-2)(3) = -2x - 6$
* $-(2y-9)$ = $-2y - (-9)$
  $= -2y + 9$

End Of Lesson 4