Proving Using Induction

1. Base Case -- show that the assumption is true for a very basic, trivial case (this is usually where the variable = 0 or 1).

2. Pick an arbitrary number (which we call k), and assume that the conclusion is true for this value.

3. Replace n with k+1, and rewrite the left side AND the right side of the equation with k+1 instead of n.

4. Figure out what your goal is (ie: simplify the right side so you know what you're trying to obtain).

5. Work on the left side, trying to get it to equal (or be less than, or whatever the situation calls for) the right side.

6. The trick to the entire proof is using the **induction assumption**. This is where you use the fact that you have assumed the result to be true for the n = k case.

$1+2+3+\cdots+100 = \frac{100(100+1)}{2}$

Classic Example:

Prove that $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

1. Base Case -- I will use several base cases just to be certain.

$n = 1$	$n = 2$	$n = 3$
$1 = 1(1+1)/2$	$1 + 2 = 2(2+1)/2$	$1 + 2 + 3 = 3(3+1)/2$
$1 = 2/2$ ✓	$3 = 6/2$ ✓	$6 = 12/2$ ✓

2. Pick an arbitrary integer k <= n, and assume $1 + 2 + 3 + \ldots + k = k(k+1)/2$

3. For n = k+1... $1 + 2 + 3 + \ldots + (k+1) = \frac{(k+1)((k+1)+1)}{2}$

4. $1 + 2 + 3 + \ldots + (k+1) \stackrel{?}{=} \frac{(k+1)(k+2)}{2}$ ← What we want!

5. $\underbrace{1 + 2 + 3 + \ldots k}_{\frac{k(k+1)}{2}} + (k+1)$ crucial step!

by induction assumption

6. $\frac{k(k+1)}{2} + (k+1)$

$\frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

$\frac{k(k+1) + 2(k+1)}{2}$

$\frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)}{2}$

This is what we wanted!

Therefore, because we proved that the theorem was true for all k <= n, and we proved that, if it is true for k, then it is true for k+1, then it must be true for all integers >= the smallest base case found to be true.

Prove that $2^n < n!$ for $n>=4$.

Base cases:

$2^4 < 4!$
$2*2*2*2 < 4*3*2*1$
$16 < 24$

$2^5 < 5!$
$2*2*2*2*2 < 5*4*3*2*1$
$32 < 120$

Pick arbitrary integer k so that k <= n, and assume that $2^k < k!$

We want:
$< (k+1)!$
$< k!(k+1)$

For n = k+1...

$2^{k+1} \overset{?}{<} (k+1)!$
$2*2^k$

< k! by induction assumption

$2^{k+1} = 2*2^k < 2*k! < (k+1)*k!$

if k >= 4   if k >= 2

Therefore, because we proved that the theorem was true for all k <= n, and we proved that, if it is true for k, then it is true for k+1, then it must be true for all integers >= 4.

Prove that $5^n - 1$ is divisible by 4 for n >= 1

Base cases: $5^1 - 1$ is divisible by 4
4 is divisible by 4
✓

$5^2 - 1$ is divisible by 4
24 is divisible by 4
✓

Pick arbitrary integer k so that k <= n, and assume that $5^k - 1$ is divisible by 4

We want:
Divisible by 4

For n = k+1... $5^{k+1} - 1$

$5*5^k - 1$

$4*5^k + 5^k - 1$

divisible by 4
by induction assumption

$4*5^k + 4*(\text{something})$

$4*(5^k + \text{something})$

$4*(\text{some stuff})$

The definition of being divisible by 4 means that it is 4*(something) -- therefore, this is divisible by 4, and we have what we want.

Therefore, because we proved that the theorem was true for all k <= n, and we proved that, if it is true for k, then it is true for k+1, then it must be true for all integers >= 1.

Prove that for all pos. integers $\overset{n}{\cancel{k}}$, $\underset{n}{\cancel{k}}^3 + 2\underset{n}{\cancel{k}}$ is a multiple of 3

Base cases:

$1^3 + 2*1$ is a multiple of 3	$2^3 + 2*2$ is a multiple of 3
$1 + 2$ is a multiple of 3	$8 + 4$ is a multiple of 3
3 is a multiple of 3 ✓	12 is a multiple of 3 ✓

Pick arbitrary integer k so that k <= n, and assume that $k^3 + 2k$ is a multiple of 3

Recall what it means for something to be a multiple of a 3.
This means that it equals 3*(something).

We want:
Multiple of 3

For n = k+1...

$$(k+1)^3 + 2(k+1)$$
$$k^3 + 3k^2 + 3k + 1 + 2k + 2$$

notice the strategic arrangement of terms

$$\underbrace{k^3 + 2k}_{\text{multiple of 3 by induction assumption}} + 3k^2 + 3k + 3$$

$$3*(\text{something}) + 3k^2 + 3k + 3$$
$$3*(\text{something}) + 3(k^2 + k + 1)$$
$$3*(\text{something} + k^2 + k + 1)$$
$$3*(\text{some stuff})$$

Therefore, because we proved that the theorem was true for all k <= n, and we proved that, if it is true for k, then it is true for k+1, then it must be true for all positive integers.