

ArsDigitaUniversity
Month2:DiscreteMathematics -ProfessorShaiSimonson

ProblemSet2 –Sets,Functions,Big -O,RatesofGrowth

1. Prove by formal logic:
 - a. The complement of the union of two sets equals the intersection of the complements.
 - b. The complement of the intersection of two sets equals the union of the complements.
 - c. $(B - A) \cup (C - A) = (B \cup C) - A$.
 - d. If two sets are subsets of each other then they are equal.

2. Generalize De Morgan's laws for n sets and prove the laws by induction.

3. Prove by induction on the size of the set, that the powerset $P(A)$ has cardinality $2^{|A|}$.

4. $A \oplus B$ is defined to be the set of all elements in A or B but not in both A and B .
 - a. Determine whether or not \oplus is commutative. Prove your answer.
 - b. Determine whether \oplus is associative. Prove your answer.
 - c. Determine whether \oplus can be distributed over union. Prove your answer.
 - d. Determine whether \oplus can be distributed over intersection. Prove your answer.

5. Assume a universal set of 8 elements. Given a set $A = a_7 a_6 a_5 a_4 a_3 a_2 a_1 a_0$, represented by 8 bits, explain how to use bitwise and/or not operations, in order to:
 - a. Extract the rightmost bit of set A .
 - b. Make the odd-numbered bits equal 0.
 - c. Make bits 4-6 equal to 1.
 Given another set B , explain how to:
 - a. Determine if $A \subseteq B$.
 - b. Extract $A - B$.

6. The Inclusion/Exclusion Theorem for three sets says that if there are three sets, A, B and C , then $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. Note that the converse of this theorem is not true. Given a set of 29 students, where 8 need housing and financial aid; 12 need housing, financial aid, and an e-mail account; 17 need an e-mail account and financial aid; 23 need housing, 20 need financial aid, 19 need an e-mail account, and 4 students don't need anything.
 - a. How many students need both housing and an e-mail?
 - b. Change the numbers to 57, 8, 3, 21, 21, 32, 31, 8. What is your answer now?
 - c. Which answer is bogus and why?

7. How many numbers are there between 1 and 10000, which are either even, end in 0, or have the sum of its digits divisible by 9? Hint: Use inclusion/exclusion.

8. Prove that $f(x) = x^3 - 1000x^2 + x - 1$ is $\Omega(x^3)$ and $O(x^3)$.

9. Prove that $1^k + 2^k + 3^k + \dots + n^k$ is $\Omega(n^{k+1})$ and $O(n^{k+1})$, for any fixed $k > 0$.

10. Order the following functions in order of their growth rate. If $f(x)$ is $O(g(x))$, but $g(x)$ is not $O(f(x))$, then put $f(x)$ above $g(x)$. If they are each big- O of each other, then place them on the same level.

$x^2 + x^3$ 3^x $x!$ $x/\log_2 x$ $x^2 + 2^x$ $x \log_2 x$ $2^{x \log x}$ $\log x^2$
 $\log_2 x!$ $\log_2 x, \ln x,$ 2^x $x(1+2+\dots+x)$ $\log \log x$ 2^{x^2} $\log^2 x$

11. Compute the sum of the infinite series below.

- a. $1 + 1/4 + 1/16 + 1/64 + \dots$
- b. $1 + 2/4 + 3/16 + 4/64 + \dots$

12. Telescoping Series.

- a. Using the identity $1/(k(k+1)) = 1/k - 1/(k+1)$, find the value of the infinite sum $1/(1*2) + 1/(2*3) + 1/(3*4) + \dots$
- b. The n th triangular number is defined to be the sum of the first n positive integers. What is the value of the infinite sum of the reciprocals of the triangular numbers.

13. Countability Proofs.

- a. Prove that the positive odd numbers have the same cardinality as the positive even numbers.
- b. Prove that the set of all integer points in the positive quadrant of 3-dimensional space are countable.
- c. Prove by induction that the set of all vectors in k dimensions with positive integer values is countable.
- d. Prove that the number of Scheme programs is countable. Hint: Order them by the number of characters each contains.

14. The following is a version of Russell's Paradox for sets, described in your text (Rosen) in problem 26, on page 45. Consider any computer program that takes other programs as input, and outputs *yes* or *no* based on some criterion. (A compiler or interpreter, for example). It is possible for such a program to be fed back into itself, and depending on the program it might either say *yes* (a Scheme interpreter written in Scheme) or *no* (a Scheme interpreter written in Java). Call the programs that say *no* to themselves *self-hating* programs. Now assume that there is a computer program that takes other programs as input and says *yes* to all the *self-hating* programs, and *no* otherwise.

- a. Assuming this program never runs forever on an input, analyze what happens when this program is input to itself. Is such a program possible?
- b. Consider the possibility that the program will run forever answering neither *yes* nor *no* on some input, and reanalyze the paradox. Is such a program possible?

15. Counting each arithmetic calculation or comparison, extraction or exchange of a card as one operation, what is the worst-case order of growth of an algorithm that sorts numbered cards in the following way?

- a. Find the largest valued card in the deck by shuffling through one card at a time extracting a card if it is the largest one seen so far, and swapping the previously largest card back into the deck. When the largest has been found, place this card facedown in a new pile and repeat the previous process until no cards in the original pile are left. Explain your answer.
- b. This time we assume that the largest number on any of the n cards is n^2 . We sort the cards by placing a set of n^2 plates numbered from 1 to n^2 on a table. Then one by one, place each card on top of the numbered plate equal to it on the desk. The sorted list can be extracted by looking through the piles on all n^2 plates in order.
- c. This method can be improved to work in linear time. Explain how. Hint: This is not easy. Use division, to try to turn each number into a pair of numbers, each with a value between 1 and n .