ArsDigita University

Month 2: Discrete Mathematics - Professor Shai Simonson

Problem Set 1 – ANSWERS

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(1) Logic Proofs

(a) Prove that $a \to b$ is equivalent to $\neg b \to \neg a$ using a truth table.
ANSWER: We compute the truth table values and observe they are the same. Note that $\neg b \to \neg a$ is called the contrapositive of $a \to b$. (Check out http://logik.phl.univie.ac.at/∼chris/formular-uk-zentral.html for an automated truth-table generator)

a	b	$\neg a$	$\neg b$	$a \to b$	$(\neg b) \to (\neg a)$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

(b) Prove it using *algebraic* identities.
ANSWER: Using the equivalence rules on pages 17-18 of Rosen,

$$\begin{aligned} a \to b &\Leftrightarrow \neg a \wedge b \\ &\Leftrightarrow b \wedge \neg a \\ &\Leftrightarrow (\neg\neg b) \wedge \neg a \\ &\Leftrightarrow \neg b \to \neg a \end{aligned}$$

(c) Prove that $a \to b$ is not equivalent to $b \to a$.
ANSWER: This is the sort of thing that is easy to show with a truth table (semantically) and is extremely difficult to show using the algebraic identities (syntactically). If $a \to b$ were equivalent to $b \to a$, then the two would have to agree for all possible *true/false* assignments of a and b. But this is not the case: if we let a be *true* and b be *false*, then $a \to b$ evaluates to *false*, and $b \to a$ evaluates to *true*. So the two formulas are not equivalent.

(2) Aristotle's Proof that the Square Root of Two is Irrational.

(a) Prove the *lemma,* used by Aristotle in his proof, which says that if n^2 is even, so is n. (Hint: Remember that $a \to b$ is equivalent to $\neg b \to \neg a$.
ANSWER: It is easiest to prove the contrapositive of the lemma, that if n is odd, then n^2 is odd. An odd number is one that can be written in the form of $2k + 1$ for some other whole number k. So suppose that $n = 2k + 1$. Then

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

And so n^2 is odd. This proves the contrapositive and hence the lemma.

(b) Prove that the square root of 3 is irrational using Aristotle's techniques. Make sure to prove the appropriate lemma.
ANSWER: We need the following lemma:

Lemma:
For any whole number n, n^2 is divisible by 3 implies that n is divisible by 3

Proof of Lemma: Again it is easier to prove the contrapositive:

n is not divisible by 3 implies that n^2 is not divisible by 3.

So assume that n is not divisible by 3 then n is of the form $n = 3k+1$, or $n = 3k+2$ for some whole number k. Then in the first case

$$n^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$$

which is not divisible by 3. The second case is similar. Q.E.D. Lemma.

Proof that the square root of 3 is irrational:
Suppose not, and that we can write

$$\sqrt{3} = \frac{p}{q}$$

with $\frac{p}{q}$ in lowest terms. Then

$$3q^2 = p^2$$

And so p^2 is divisible by 3. By the lemma it follows that p is divisible by 3, say $p = 3p_1$ for some whole number p_1. Plugging back in to the formula, we get

$$q^2 = 3p_1^2$$

Now the tables are turned! q^2 is divisible by 3 and so by the lemma q is divisible by 3, say $q = 3q_1$ for some whole number q_1. This implies that

$$\frac{p}{q} = \frac{3p_1}{3q_1} = \frac{p_1}{q_1}$$

This is the contradiction we were looking for: We assumed that $\frac{p}{q}$ was in lowest terms, which we can always do with a fraction, but the assumption that $\frac{p}{q} = \sqrt{3}$ implies that $\frac{p}{q}$ is *not* in lowest terms. Contradiction! Therefore $\sqrt{3}$ is not a rational number. Which is quite a disturbing thing, if you're an ancient Greek.

(c) If we use Aristotle's technique to *prove* the untrue assertion that the square root of 4 is irrational, where *exactly* is the hole in the proof?
ANSWER: The problem is that when we reach the equation

$$q^2 = 4p_1^2$$

We *cannot* use the fact that 4 divides q^2 to conclude that 4 divides q. This is just false. For example: $2^2 = 4$, or $6^2 = 36$.

(d) Using the fact that the square root of two is irrational, prove that $\sin(\frac{\pi}{4})$ is irrational.
ANSWER: There is no way that $\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$ could be equal to a rational number $\frac{p}{q}$ for then it would follow that $\sqrt{2}$ would equal the inverse, $\frac{q}{p}$, which we have shown to be impossible.

(3) In ADU-ball, you can score 11 points for a goal, and 7 for a near miss.

(a) Write a Scheme program that prints out the number of goals and the number of near misses to achieve a given total greater than 60.

(b) Prove that you can achieve any score greater than 60. Think inductively and experiment.

ANSWER:

(a) ADU-BALL returns a list of all goal, near-miss pairs that add up to the given number The loop subtracts off multiples of 7 from the given number, and adds a solution to the list if the result is divisible by 11. Note the use of let*, which unlike let, does perform each assignment before going to the next one.

```
(define (adu-ball n)
   (define (iter i result)
    (let* ((a (- n (* 7 i))) (b (remainder a 11)))
     (cond ((< a 0) result)
           ((= b 0) (iter (inc i) (cons (cons (/ a 11) i) result)))
           (else (iter (inc i) result)))))
   (iter 0 nil))

(adu-ball 187)
  ;Value: ((3 . 22) (10 . 11) (17 . 0))
```

(b) Actually you can achieve any score greater than or equal to 60. We will prove this by induction. There will actually be seven base cases here, since the induction will go down by steps of seven. Here is a formal statement of what will prove:
For any $n \geq 60$ there are natural numbers x, y such that $n = 11x + 7y$
Base Cases:

- $n = 60$. We can let $x = 1$, $y = 7$
- $n = 61$. We can let $x = 3$, $y = 4$
- $n = 62$. We can let $x = 5$, $y = 1$
- $n = 63$. We can let $x = 0$, $y = 9$
- $n = 64$. We can let $x = 2$, $y = 6$
- $n = 65$. We can let $x = 4$, $y = 3$
- $n = 66$. We can let $x = 6$, $y = 0$

Induction step. Suppose $67 \leq n$ Suppose that we can express any number k $60 \leq k < n$. We would like to show that there is a u and a v such that $n = 11u + 7v$. This isn't so bad. We know that $60 \leq n - 7 < n$, so by induction. We can find an x and a y such that.

$$11x + 7y = n - 7$$

This implies that

$$11x + 7(y + 1) = n$$

and so we can let $u = x$ and $v = y + 1$ and our induction step is proven. Q.E.D.

(4) Prove by induction that there are 2^n possible rows in a truth table with n variables.
ANSWER:
Base Case: When $n = 1$, we have 2 rows, which equals 2^1
Induction Step: Assume that an n-variable truth table has 2^n rows. We want to show that an $(n+1)$-variable truth table has 2^{n+1} rows. Well the way we construct a $(n + 1)$ variable truth-table from an n-variable truth table is we take two copies of it and we set the new variable equal to 1 in the first copy and we set the new variable equal to 0 in the second copy. So the total number of rows is

$$\begin{aligned} \text{number of rows in copy-1} + \text{number of rows in copy-2} &= 2^n + 2^n \\ &= 2^{n+1} \end{aligned}$$

So our induction step is proven. Q.E.D.

(5) In the restroom of a fancy Italian restaurant in Mansfield, MA, there is a sign which reads: *Please do not leave valuables or laptop computers in your car.*
Assuming that a laptop computer is considered a valuable, prove using formal logic, that the sentence *Please do not leave valuables in your car* is equivalent to the sign in the restroom. Prove that *Please do not leave laptops in your car* is not equivalent.
ANSWER:
This is a bit confusing at first glance, since our logical system isn't designed to deal with imperatives. If we ignore that little difficulty, there are many ways to turn the above sentences into formulas. Here is one: Let V be the statement there are valuables in the car. Let L be the statement there are laptops in the car. We know that

$$L \to V$$

We want to show that the above assumption implies that

$$L \vee V \Leftrightarrow V$$

The easiest way is to compare the truth-table values of $L \vee V$ and V in the three cases where $L \to V$ holds

L	V	$L \to V$	$L \vee V$
0	0	1	0
0	1	1	1
1	1	1	1

We see that in the cases where $L \to V$ holds, V and $L \vee V$ are equivalent to each other and not equivalent to L

(6) Prove that $a|b$, a nand b, which is defined to be $\neg(a \wedge b)$, is complete. Write $(\ a\ \to\ b)\ \to\ b$ using just | (nand), then using just ↓ (nor).
ANSWER:
(a) We just need to construct not, and from nand. Negation is very easy:

$$a|a = \overline{a \cdot a}, \text{ which is equivalent to } \overline{a}$$

Now $a \cdot b$ is equivalent to $\overline{a|b}$, which by the above is equivalent to

$$(a|b)|(a|b)$$

Therefore nand is complete, since or, and are complete.

(b) One way to do this is to simplify

$$(a \rightarrow b) \rightarrow b$$

Plugging in the definition of implication, we get,

$$\overline{(\overline{a} + b)} + b$$

Which is equivalent to

$$a \cdot \overline{b} + b$$

Distributing the or over the and, this is equivalent to

$$(a + b)(\overline{b} + b)$$

Which is equivalent to

$$(a + b)$$

To express this in terms of nand, we first express it in terms of not, and. The formula is equivalent to

$$\overline{\overline{a}\overline{b}}$$

In other words,

$$\overline{a}|\overline{b}$$

or

$$(a|a) \,|\, (b|b)$$

(c) We know that $\overline{a}$ is equivalent to $a \downarrow a$. As in the previous case, we can work with the simplified form $a + b$. We know that this is equivalent to

$$\overline{a \downarrow b}$$

by the definition of $\downarrow$. Therefore, $a + b$ is equivalent to

$$(a \downarrow b) \downarrow (a \downarrow b)$$

(7) Show how to use a truth table in order to construct a conjunctive normal form for any Boolean formula W. Hint: Consider the disjunctive normal form for $\neg W$.

ANSWER:

The method is completely dual to the method shown for getting the DNF of a formula. Let's recall how we get a DNF from the truth table values of an expression W: We pick all rows where W is true. To get the DNF, we or together one term for each of those rows. The term is constructed from the values of the variables take on in that row. For example if in that row the variables have values $x = 0$, $y = 1$, $z = 1$, then the term from that row will be $\overline{x}yz$.

To construct a CNF, we simply do the opposite at every stage of the DNF construction: We pick all rows where W is false. To get the CNF, we and together one term for each of those rows. The term is constructed from the values of the variables take on in that row. For example if in that row the variables have values $x = 0$, $y = 1$, $z = 1$, then the term from that row will be $(x + \overline{y} + \overline{z})$.

(8) Euclid proved that there are an infinite number of primes, by assuming that n is the highest prime, and exhibiting a number that he proved must either be prime itself, or else have a prime factor greater than n. Write a scheme program to find the smallest n for which Euclid's proof does *not* provide an actual prime number.

ANSWER: There was some confusion as to whether the question wants you i) to use all primes less than a given prime n, *i.e.* to generate the primes in some other fashion an then apply Euclid's proof to an initial sequence of primes or ii) to use *all primes generated by Euclid's proof.* These are not the same since Euclid's proof skips over many primes. For example, starting with 2, Euclid's proof generates 3, 7 for the next two primes, skipping over 5. The program we include here uses interpretation i).

```
(define (prime? x)    ; tests whether x is prime
   (define (iter count)
    (cond   ((> (* count count) x) true)
            ((= 0 (remainder x count)) false)
            (else (iter (inc count)))))
   (iter 2))

(define ones (cons-stream 1 ones))   ; 1,1,1,1,1,1,...

(define integers (cons-stream 1 (add-streams ones integers)))
  ;1,2,3,4,....

(define primes (stream-filter prime? integers))  ; 2,3,5,7,11,...

(define (mul-first-n-primes n) ; returns the product of the first
   (define (iter i answer)       ; n primes
    (cond ((= i 0) answer)
          (else (iter (dec i) (* (stream-ref primes i) answer)))))
   (iter n 1))

; returns first composite number of the form p1 p2 ... pi + 1
(define (find-first-composite)
   (define (iter i)
    (cond ((prime? (inc (mul-primes i)))
           (iter (inc i)))
          (else (display i)
                (display "   ")
                (inc (mul-first-n-primes i)))))
   (iter 2))

(find-first-composite)
  6
  ;Value: 30031
```

(9) You have proved before that a truth table with n variables has 2^n rows.

(a) How many different Boolean functions with n variables are there?
ANSWER: 2^{2^n}

(b) For $n = 2$, list all the functions and identify as many as you can by name.
ANSWER:

a	b	0	$a \wedge b$	$\overline{a \to b}$	a	$\overline{b \to a}$	b	$a \oplus b$	$a \vee b$	$a \downarrow b$	$a \leftrightarrow b$	$\overline{b}$	$b \to a$	$\overline{a}$	$a \to b$	$a \mid b$	1
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

(Aside: Note that $\oplus$ is just the negation of $\leftrightarrow$.)

(10) Prove by induction that for $n > 5$, $2^n > n^2$.
ANSWER: Actually, we can take $n \geq 5$.
Base Case: $n = 5$. Then $2^5 = 32$, which is greater than 5^2, which is 25

Inductive step: Suppose for a fixed n that $2^n > n^2$ We want to show that $2^{n+1} > (n+1)^2$. Multiplying the n-th case by 2, we get

$$2 \cdot 2^n > 2n^2,$$

or

$$2^{n+1} > 2n^2.$$

So all we have to show is that

$$2n^2 > (n+1)^2$$

for all $n \geq 5$. Expanding out, this is true if and only if

$$n^2 - 2n - 1 > 0$$

for all $n \geq 5$. This in turn could be proven by induction or we could use the following trick. If we add 2 to the left hand side it becomes a perfect square:

$$n^2 - 2n + 1 > 2$$

In other words,

$$(n+1)^2 > 2$$

But this is true if and only if

$$(n+1) > \sqrt{2}$$

or equivalently

$$n > \sqrt{2} - 1 \approx 0.4$$

So it is certainly true that

$$2n^2 > (n+1)^2$$

for all $n \geq 5$ and so

$$2^{n+1} > 2n^2 > (n+1)^2$$

and we have proved the inductive step. Q.E.D.

(11) Guess the number of different ways for n people to arrange themselves in a straight line, and prove your guess is correct by induction.

ANSWER: The number of ways for n people to arrange themselves is $n!$. We will see why a bit later, but all we have to do here is prove the formula by induction.

Base Case: The number of ways one person can arrange themselves in line is 1, which agrees with 1!

Inductive Step: Assume that for a given n the number of ways n people can arrange themselves in line is n. We would like to show that the number of ways $n+1$ people can arrange themselves in line is then necessarily $(n+1)!$. We can do this by breaking the process in to two steps. First we line up the first n people and then we pick a spot for the last person to go to. By the rule of independent outcomes (which we assume you just sort of know about from common sense - we'll see it again when we do probability) the number of ways to do both things in sequence is the product of the number of ways to do each separate thing. Lining up n people can be done in $n!$ ways by the induction hypothesis. The $(n+1)^{\text{st}}$ has $n+1$ places to go - either before all the people, or after the first person, or after the second person .... (Technically, we should also prove that there are $n+1$ places to go by another induction, but we'll skip this for our sanity and since the claim is sufficiently believable). Therefore the number of ways to arrange $n+1$ people is the product of the two numbers,

$$(n+1) \cdot n! = (n+1)!$$

And so the inductive step isproven. Q.E.D.

(12) Use logic with quantifiers and predicates to model the following three statements:
- All students are taking classes.
- Some students are not motivated.
- Some people taking classes are not motivated.

Prove, using resolution methods, that the third statement follows logically from the first two. (Reminder: You must take the conjunction of the first two statements and the negation of the third, and derive a contradiction.)

ANSWER:

(a) The universe is the set of all people. We need the following predicates

$$S(x) \text{ if and only if } x \text{ is a student}$$

$$M(x) \text{ if and only if } x \text{ is motivated}$$

$$C(x) \text{ if and only if } x \text{ is taking classes}$$

The sentences translate to

$$\forall x S(x) \rightarrow C(x)$$

$$\exists x S(x) \wedge \neg M(x)$$

$$\exists x C(x) \wedge \neg M(x)$$

(b) Let's prove that the third follows from the first two. We'll go about it like the hint says, by assuming the first two and the negation of the third (we don't really have to take the conjunction of the first two statements, we'll just list them separately in our proof)

1.	$\forall x S(x) \to C(x)$	
2.	$\exists x S(x) \wedge \neg M(x)$	
3.	$\neg \exists x C(x) \wedge \neg M(x)$	1.-3. are the assumptions we are making
4.	$\forall x \neg C(x) \vee M(x)$	by 3. and the properties of quantifiers
5.	$S(p) \wedge \neg M(p)$	for some p, by existential instantiation of 2.
6.	$S(p) \to C(p)$	by universal instatiation of 1.
7.	$\neg S(p) \vee C(p)$	by the definition of $\to$
8.	$\neg C(p) \vee M(p)$	by universal instatiation of 4.
9.	$\neg S(p) \vee M(p)$	by resolution of 7. and 8.
10.	$\neg(\neg S(p) \vee M(p))$	by 5. and Demorgan's Laws

This is a contradiction, since 9. and 10. are negations of each other. Therefore if 1. and 2. hold, 3. cannot hold. Q.E.D.

(13) The following algebraic idea is central for Karnaugh maps. Karnaugh maps are a method of minimizing the size of circuits for digital logic design.

(a) Using algebraic manipulation, prove that the two Boolean formulae below are equivalent. (Hint: $x(a + \overline{a})$ is equivalent to x.)

$$x\overline{y} + y\overline{z} + \overline{x}z \qquad \text{and} \qquad \overline{x}y + \overline{y}z + x\overline{z}$$

ANSWER: We'll show that the two formulas have the same canonical disjunctive normal form (CDNF). The formulas are already in disjunctive normal form. The CDNF is the DNF you would get using a truth table. What distinguishes it from DNF is that each variable in the formula must appear in each of the terms being or'ed togther. The formulas above do not include all of the variables x, y, z in each term. We can fix by using the trick suggested by the hint. The first formula expands to

$$\begin{aligned} x\overline{y} + y\overline{z} + \overline{x}z &\Leftrightarrow x\overline{y}(z + \overline{z}) + (x + \overline{x})y\overline{z} + \overline{x}(y + \overline{y})z \\ &\Leftrightarrow x\overline{y}z + x\overline{y}\overline{z} + xy\overline{z} + \overline{x}y\overline{z} + \overline{x}yz + \overline{x}\overline{y}z \end{aligned}$$

The second formula expands to

$$\begin{aligned} \overline{x}y + \overline{y}z + x\overline{z} &\Leftrightarrow \overline{x}y(z + \overline{z}) + (x + \overline{x})\overline{y}z + x(y + \overline{y})\overline{z} \\ &\Leftrightarrow \overline{x}yz + \overline{x}y\overline{z} + x\overline{y}z + \overline{x}\overline{y}z + xy\overline{z} + x\overline{y}\overline{z} \end{aligned}$$

Now the two expanded formulas are the same up to permutation, so the original formulas are equivalent.

(b) Verify your results using a truth table.
ANSWER:

x	y	z	$x\overline{y} + y\overline{z} + \overline{x}z$	$\overline{x}y + \overline{y}z + x\overline{z}$
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
1	0	0	1	1
0	1	1	1	1
1	0	1	1	1
1	1	0	1	1
1	1	0	0	0

Yes!

(14) The exclusive-or operator $\oplus$, is defined by the rule that $a \oplus b$ is true whenever a or b is true but not both.

(a) Calculate $x \oplus x$, $x \oplus \neg x$, $x \oplus 1$, $x \oplus 0$.
ANSWER: 0, 1, $\neg x$, x

(b) Prove or disprove that $x + y \oplus z = (x + y) \oplus (x + z)$
ANSWER: This is not true. If we let x be true, the left-hand side will always be true, whereas the right-hand side will always be false.

(c) Prove or disprove that $x \oplus (y + z) = (x \oplus y) + (x \oplus z)$
ANSWER: This is not true. If we let x be true, the left-hand side is equivalent to $\neg(y + z)$, *i.e.* nor of y, z, and the right hand is equivalent to $(\neg y) \vee (\neg z)$, ı.e nand of y, z. And these are definitely not equivalent.

(d) Write conjunctive normal form and disjunctive normal form formulae for $x \oplus y$
ANSWER:

$$\text{CNF:}(a + b)(\overline{a} + \overline{b})$$
$$\text{DNF:}\overline{a}b + a\overline{b}$$

(e) The exclusive-or operator is not *complete*. Which of the three operators {and, or, not} can be combined with exclusive-or to make a *complete* set.
ANSWER:

- $\oplus$ and $\neg$ are not complete. This is not the most straightforward thing in the world to show. If they were complete, we would be able to generate any four-bit sequence (column of a truth table) with the operators just starting from 0011, 0101. The problem is that both of the starting sequences have an even number of ones, and $\oplus$, $\neg$ will keep the number of ones even, so we will never be able to generate 1000, for example.
- $\oplus$ and $\vee$ are not complete. If they were complete, we would be able to generate any four-bit sequence (column of a truth table) with the operators just starting from 0011, 0101. However both $\oplus$ and $\vee$ cannot change the 0 in the first positions of 0011, 0101 to a 1, so we will never be able to generate any sequence starting with a 1.
- $\oplus$ and $\wedge$ are not complete for the same reason as the previous case.

(15) The n^{th} triangle number T_n is defined to be the sum of the first n integers.

(a) Prove by induction that $T_n = n(n+1)/2$.
ANSWER:
Base case: $T_1 = 1 = 1(1+1)/2$
Inductive Step: Suppose that for a fixed n, it is true that $T_n = \frac{n(n+1)}{2}$. We need to show that it follows necessarily that $T_{n+1} = \frac{(n+1)(n+2)}{2}$. We can do this by expressing T_{n+1} in terms of T_n and manipulating the result

$$\begin{aligned} T_{n+1} = T_n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ = (n+1)(\frac{n}{2}+1) &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

Therefore the inductive step holds and it follows that $T(n) = \frac{n(n+1)}{2}$ for every n.

(b) Prove algebraically using (a), that

$$n^3 + (1+2+\cdots+(n-1))^2 = (1+2+\cdots(n-1)+n)^2$$

ANSWER: We want to show that

$$\begin{aligned} n^3 &= (1+2+\cdots+n)^2 - (1+2+\cdots+(n-1))^2 \\ &= T_n^2 - T_{n-1}^2 \end{aligned}$$

Using the formula for T_n, the right hand-side becomes

$$\begin{aligned} \frac{n^2(n+1)^2}{4} - \frac{(n-1)^2n^2}{4} &= \frac{n^2}{4}((n+1)^2 - (n-1)^2) \\ &= \frac{n^2}{4}(4n) \\ &= n^3 \end{aligned}$$

Q.E.D.

(c) Using (b) guess a formula for

$$C_n = 1^3 + 2^3 + 3^3 + \cdots + n^3,$$

and prove it by induction.
ANSWER: This our first encounter with a telescoping series (*c.f.* problem set 2)

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + n^3 &= (T_1^2 - T_0^2) + (T_2^2 - T_1^2) + \cdots + (T_n^2 - T_{n-1}^2) \\ &= (-T_0^2 + T_n^2) \\ &= T_n^2 = \frac{n^2(n+1)^2}{4} \end{aligned}$$

Even though it is fairly straightforward from the above, we'll prove that

$$C_n = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

by induction, just for practice.
Base case: $C_1 = 1^3 = 1$, and $\frac{1^2 2^2}{4} = 1$
Inductive Step: Suppose that for a fixed n, that $C_n = \frac{n^2(n+1)^2}{4}$. We want to show

that this necessarily implies $C_{n+1} = \frac{(n+1)^2(n+2)^2}{4}$. We start by expressing C_{n+1} in terms of C_n and then manipulate the result:

$$\begin{aligned} C_{n+1} &= C_n + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= (n+1)^2(\frac{n^2}{4} + (n+1)) \\ &= (n+1)^2\frac{n^2+4n+4}{4} \\ &= (n+1)^2\frac{(n+2)^2}{4}. \end{aligned}$$

This is what we wanted to show, so the inductive step is proven.
Q.E.D.

(16) Guess a formula for the sum below, and prove you are right by induction.

$$1 + 1(2) + 2(3) + 3(4) + \cdots + n(n+1)$$

ANSWER: The 1 at the beginning of the formula doesn't fit the pattern of the tail of the sum, so we'll figure a formula for

$$A_n := 1(2) + 2(3) + 3(4) + \cdots + n(n+1)$$

and then we can add 1 to it at the end. We can reduce this sum into two more basic sums:

$$\begin{aligned} 1(2) + 2(3) + 3(4) + \cdots + n(n+1) &= 1(1+1) + 2(2+1) + \cdots + n(n+1) \\ &= (1^2 + 2^2 + \cdots n^2) + (1 + 2 + \cdots n) \end{aligned}$$

The second sum is one we already have a closed form for. I claim that the formula for the first sum is

$$(1^2 + 2^2 + \cdots n^2) = \frac{n(n+1)(2n+1)}{6},$$

so the original sum is

$$\begin{aligned} A_n := 1 + 1(2) + 2(3) + 3(4) + \cdots + n(n+1) &= 1 + \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\ &= 1 + \frac{n(n+1)}{2}(\frac{2n+1}{3} + 1) \\ &= 1 + \frac{n(n+1)(2n+4)}{6}. \end{aligned}$$

The beauty of proof-by-induction is you don't have to know at all how I came up with the formula for the sum. You can still use induction to prove that it is correct. Let's do that now:
Base Case: $A_0 = 1$, $1 + \frac{0(0+1)(2\cdot 0+4)}{6} = 1$
Inductive Step: Suppose that for a fixed n, $A_n = 1 + \frac{n(n+1)(2n+4)}{6}$ we want to show

that it follows necessarily that $A_{n+1} = 1 + \frac{(n+1)(n+2)(2n+6)}{6}$. We start by expressing A_{n+1} in terms of A_n and then manipulate the result:

$$\begin{aligned} A_{n+1} &= A_n + (n+1)(n+2) \\ &= 1 + \frac{n(n+1)(2n+4)}{6} + (n+1)(n+2) \\ &= 1 + (n+1)(\frac{n(2n+4)}{6} + (n+2)) \\ &= 1 + (n+1)(\frac{2n^2+10n+12}{6}) \\ &= 1 + (n+1)\frac{(n+2)(2n+6)}{6}. \end{aligned}$$

This is what we wanted to show, so the inductive step is proven.
Q.E.D.