ArsDigita University
Month 2: Discrete Mathematics - Professor Shai Simonson

Problem Set 1 – Logic, Proofs, and Mathematical Reasoning

1. Logic Proofs.
   a. Prove that $a \rightarrow b$ is equivalent to $\neg b \rightarrow \neg a$ using a truth table.
   b. Prove it using *algebraic* identities.
   c. Prove that $a \rightarrow b$ is not equivalent to $b \rightarrow a$.

2. Aristotle's Proof that the Square Root of Two is Irrational.
   a. Prove the *lemma*, used by Aristotle in his proof, which says that if n^2 is even, so is *n*. (Hint: Remember that $a \rightarrow b$ is equivalent to $\neg b \rightarrow \neg a$).
   b. Prove that the square root of 3 is irrational using Aristotle's techniques. Make sure to prove the appropriate lemma.
   c. If we use Aristotle's technique to *prove* the untrue assertion that the square root of 4 is irrational, where *exactly* is the hole in the proof?
   d. Using the fact that the square root of two is irrational, prove that sin (π/4) is irrational.

3. In ADU-ball, you can score 11 points for a goal, and 7 for a near miss.
   a. Write a Scheme program that prints out the number of goals and the number of near misses to achieve a given total greater than 60.
   b. Prove that you can achieve any score greater than 60. Think inductively and experiment.

4. Prove by induction that there are 2^n possible rows in a truth table with *n* variables.

5. In the restroom of a fancy Italian restaurant in Mansfield, MA, there is a sign that reads: *Please do not leave valuables or laptop computers in your car.* Assuming that a laptop computer is considered a valuable, prove using formal logic, that the sentence *Please do not leave valuables in your car* is equivalent to the sign in the restroom. Prove that *Please do not leave laptops in your car* is not equivalent.

6. Prove that $a \mid b$, (a nand b), which is defined to be $\neg(a \wedge b)$, is complete. Write $(a \rightarrow b) \rightarrow b$ using just | (nand), then using just $\downarrow$ (nor).

7. Show how to use a truth table in order to construct a conjunctive normal form for any Boolean formula *W*. Hint: Consider the disjunctive normal form for $\neg W$.

8. Euclid proved that there are an infinite number of primes, by assuming that *n* is the highest prime, and exhibiting a number that he proved must either be prime itself, or else have a prime factor greater than *n*. Write a scheme program to find the smallest *n* for which Euclid's proof does *not* provide an actual prime number.

9. You have proved before that a truth table with n variables has 2^n rows.

   a. How many different Boolean functions with n variables are there?
   b. For $n=2$, list all the functions and identify as many as you can by name.

10. Prove by induction that for $n>4$, $2^n>n^2$.

11. Guess the number of different ways for n people to arrange themselves in a straight line, and prove your guess is correct by induction.

12. Use logic with quantifiers and predicates to model the following three statements:

    All students are taking classes. Some students are not motivated. Some people taking classes are not motivated.

    Prove, using resolution methods, that the third statement follows logically from the first two. (Reminder: You must take the conjunction of the first two statements and the negation of the third, and derive a contradiction.)

13. The following algebraic idea is central for Karnaugh maps. Karnaugh maps are a method of minimizing the size of circuits for digital logic design.

    a. Using algebraic manipulation, prove that the two Boolean formulae below are equivalent. (Hint: $x(a+\neg a)$ is equivalent to x.)
       $\neg yx + \neg zy + \neg xz$ and $\neg xy + \neg yz + \neg zx$

    b. Verify your results using a truth table.

14. The exclusive-or operator $\oplus$, is defined by the rule that $a \oplus b$ is true whenever a or b is true but not both.

    a. Calculate $x \oplus x$, $x \oplus \neg x$, $x \oplus 1$, $x \oplus 0$.
    b. Prove or disprove that $x + (y \oplus z) = (x + y) \oplus (x + z)$
    c. Prove or disprove that $x \oplus (y + z) = (x \oplus y) + (x \oplus z)$
    d. Write conjunctive normal form and disjunctive normal form formulae for $x \oplus y$
    e. The exclusive-or operator is not *complete*. Which ones, if any, of the three operators {and, or, not} can be combined with exclusive-or to make a *complete* set.

15. The nth triangle number T_n is defined to be the sum of the first n integers.
    a. Prove by induction that $T_n = n(n+1)/2$.
    b. Prove algebraically using (a), that $n^3 + (1+2+\ldots+(n-1))^2 = (1+2+\ldots(n-1) + n)^2$
    c. Using (b) guess a formula for $1^3 + 2^3 + 3^3 + \ldots + n^3$, and prove it by induction.

16. Guess a formula for the sum below, and prove you are right by induction.
    $1 + 1(2) + 2(3) + 3(4) + \ldots + n(n+1)$