18.02 Multivariable Calculus
Fall 2007

18.02 Problem Set 1

Due Thursday 9/13/07, 12:45 pm.

18.02 Supplementary Notes and Problems. This is where to find the exercises labelled 1A, 1B, etc.

Problem Sets have two parts, A and B.

Part A has problems from the text, with answers to many in the back of the text, and problems from the Notes with solutions at the end of the Notes. Look at the solutions if you get stuck, but try to do as much as possible without them. Hand in the underlined problems only; the others are for more practice. Part A will be graded quickly, checking that the problems are there and the solutions not merely copied.

Part B consists of unsolved problems, is worth more points, and will be graded more carefully. Many of these problems are longer multi-part exercises posed here because they do not fit conveniently into an exam or short-answer format.

Advice: Make sure that you understand the problems by comparing your answers against the solutions, whether before (Part A) or after (Part B) the assignment is due. Keep up with the work in small installments – **don't leave it all for a marathon session on Wednesday night**. You can't learn well under time pressure. To help you keep up, each problem is labelled with the day on which you will have the needed background for it.

Homework Rules: Collaboration on problem sets is encouraged, **but**

a) **Attempt each part of each problem yourself.** Read each portion of the problem before asking for help. If you don't understand what is being asked, ask for help interpreting the problem and then make an honest attempt to solve it.

b) **Write up each problem independently.** On both Part A and B exercises you are expected to write the answer in your own words.

c) **Write on your problem set whom you consulted and the sources you used.** If you fail to do so, you may be charged with plagiarism and subject to serious penalties.

d) **It is illegal to consult materials from previous semesters.**

Part A (15 points)

Hand in the underlined problems only; the others are for more practice.

(*Notation:* 12.1/17 = Book, Section 12.1, problem 17; 1A/1 = Suppl. Notes, page 1A, problem 1).

Recitation. Wed Sept. 5 Introduction to vectors: addition, scalar multiplication
Read: 12.1
Work: 12.1/ 17, 23, 45; 1A/ 1, 5, 6, 7, 8, 9, 11.

Lecture 1. Thu Sept. 6 Dot product
Read: 12.2
Work: 1B/ 1, 2, 5ab, 11, 12, 13, 14.

Lecture 2. Fri Sept. 7 Determinants, cross product
Read: Notes D, Book 12.3
Work: 1C/ 1, 2, 3, 5a, 6, 7; 1D/ 1, 2, 3, 4, 5, 7.

Lecture 3. Tue Sept. 11 Matrices and inverse matrices
Read Notes M.1, M.2 (pp. 1–7)
Work: 1F/ 5ab, 8a, 9; 1G/ 3, 4, 5.

Part B (27 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor", "the student next to me in recitation."
Optional: note which of these people or resources, if any, were particularly helpful to you.

Problem 1. (Thursday, 5 points: 2+1+1+1)

The eight vertices of a cube centered at $(0, 0, 0)$ of side length 2 are at $(\pm 1, \pm 1, \pm 1)$.

a) Find the four vertices of the cube, starting with $(1, 1, 1)$, that form a regular tetrahedron. Confirm your answer by finding the length of an edge and explaining why all edges have the same length.

b) A methane molecule consists of a hydrogen atom at each of the vertices of a regular tetrahedron and a carbon atom at the center. Find the "bond angle", i.e. the angle made by vectors from the carbon atom to two hydrogen atoms (use a calculator; round your answer).

c) Use dot product to find the angle between two adjacent edges (sharing a common vertex) of the tetrahedron; and the angle between two opposite edges. Explain your answers using symmetry.

d) (Friday) Find the area of a face of the tetrahedron.

Problem 2. (Thursday, 3 points: 1+1+1)

Consider a triangle in the plane with vertices P_1, P_2, and P_3.

a) Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ be the vectors in the plane from the points P_1, P_2 and P_3 respectively to a point P. Express in terms of the dot product and these three vectors the condition that P is on the altitude of the triangle $P_1P_2P_3$ from the vertex P_1. (By altitude we mean the entire line through a vertex perpendicular to the opposite side, not just the segment from the vertex to the side.)

b) Assume that P is at the intersection of the altitudes from P_1 and P_2. Show that $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = \boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = \boldsymbol{v}_2 \cdot \boldsymbol{v}_3$.

c) Under the assumptions in (b), show that P is also on the altitude from P_3. (Hence all three altitudes meet in one point, called the orthocenter.)

Problem 3. (Friday, 3 points)

Four vectors are erected perpendicularly to the four faces of a general tetrahedron. Each vector is pointing outwards and has length equal to the area of the face. Show that the sum of these four vectors is 0.

Hint: let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be vectors representing the three edges starting from a fixed vertex. Express each of the four vectors in terms of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$, and show that their sum is the zero vector; do not introduce a coordinate system.

Problem 4. (Tuesday, 9 points: 1+2+2+2+2)

Orthogonal matrices are matrices A that satisfy the identity $A\,A^T = I$ (I is the identity matrix). An equivalent definition of the orthogonal matrix property is that $A^T A = I$

because the left and right inverses of a square matrix are the same (see 1G-9b). The equation $AA^T = I$ says that the rows of A are perpendicular to each other and of unit length, whereas the equation $A^TA = I$ says that the columns of A are perpendicular to each other and of unit length. The geometric significance of orthogonal matrices is that multiplication by an orthogonal matrix preserves lengths of vectors and the absolute values of angles between them:

$$|A\mathbf{v}| = |\mathbf{v}| \text{ and } |\angle(A\mathbf{v}, A\mathbf{w})| = |\angle(\mathbf{v}, \mathbf{w})|.$$

There are two types of orthogonal matrices, rotations and reflections.

a) In 2-dimensional space, rotations are given by

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Find $\mathbf{u} = A_\theta \hat{\boldsymbol{\imath}}$ and $\mathbf{v} = A_\theta \hat{\boldsymbol{\jmath}}$, and draw a picture of $\mathbf{u}$ and $\mathbf{v}$ for $\theta = \pi/4$.

b) Use the addition formulas for sine and cosine to deduce that $A_{\theta_1} A_{\theta_2} = A_{\theta_1+\theta_2}$. Say in words what this matrix formula means about rotations.

c) Calculate A_θ^{-1}, and use this to verify that $A_\theta A_\theta^T = I$ (in other words, rotations are orthogonal matrices). Also verify that $A_\theta^{-1} = A_{-\theta}$, and give a geometric reason why this property holds.

d) Find the four orthogonal 2×2 matrices with first entry $a_{11} = -1/\sqrt{2}$. Hint: try different signs. (See 1F-9 and 1F-10).

e) Next to each of the matrices in your list in part (d), draw what the matrix does to the letter F in the plane. Explain how the sign of the determinant of the matrix is related to the appearance of the transformed F.

Problem 5. (Tuesday, 7 points: 1+2+2+2)

Cookies, doughnuts, and croissants contain essentially the same ingredients (flour, sugar, egg, butter) but in different proportions. For example, it takes 22 grams of flour to make a cookie, versus 40 g for a doughnut and 50 g for a croissant. The compositions of the various pastries can be encoded in a 4×3 matrix,

$$M = \begin{pmatrix} 22 & 40 & 50 \\ 18 & 10 & 3 \\ 5 & 14 & 5 \\ 10 & 10 & 22 \end{pmatrix},$$

where the entries in the i-th row represent the amounts of ingredient i ($i = 1$ for flour, 2 for sugar, 3 for egg, 4 for butter) required to make the various types of pastries.

a) Consider an assortment of x_1 cookies, x_2 doughnuts and x_3 croissants, and form the column vector

$$\boldsymbol{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

What do the entries of the vector $M\boldsymbol{X}$ correspond to?

b) Each of the four ingredients has a specific nutritional value: for example, 100 grams of flour contain 10 g of protein, 76 g of carbohydrates, and 1 g of fat. Proceeding similarly for all four ingredients, we can build a 3×4 matrix,

$$N = \begin{pmatrix} 0.10 & 0 & 0.13 & 0 \\ 0.76 & 1.00 & 0.01 & 0 \\ 0.01 & 0 & 0.10 & 0.82 \end{pmatrix},$$

where the entries in the i-th column represent the proportions of protein, carbohydrates and fat in ingredient i.

Give a matrix formula for the total nutritional value of the assortment of pastries considered in (a). (Keep your answer in symbolic form, do not evaluate numerically.)

c) Give a matrix formula expressing the numbers x_i of pastries of each type which will add up to y_1 g of protein, y_2 g of carbohydrates, and y_3 g of fat. Express your answer in the form $\boldsymbol{X} = A\boldsymbol{Y}$, and give both a formula for A and numerical values for its entries (use either a calculator or Matlab; short directions for Matlab can be found on the course web page in the Assignments section.

d) The recommended daily amounts of protein, carbohydrates and fat for a 2000 calorie diet are 50, 300, and 65 grams respectively. If you wanted to follow those guidelines while eating only cookies, doughnuts, and croissants, how many pastries of each type should you eat daily? What is wrong with your answer? Explain.