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18.02 Multivariable Calculus, Fall 2007
Transcript – Lecture 6

So, if you remember last time, we looked at parametric equations -- -- as a way of describing the motion of a point that moves in the plane or in space as a function of time of your favorite parameter that will tell you how far the motion has progressed. And, I think we did it in detail the example of the cycloid, which is the curve traced by a point on a wheel that's rolling on a flat surface.

So, we have this example where we have this wheel that's rolling on the x-axis, and we have this point on the wheel. And, as it moves around, it traces a trajectory that moves more or less like this. OK, so I'm trying a new color. Is this visible from the back? So, no more blue. OK, so remember, in general, we are trying to find the position, so, x of t, y of t, maybe z of t if we are in space -- -- of a moving point along a trajectory.

And, one way to think about this is in terms of the position vector. So, position vector is just the vector whose components are coordinates of a point, OK, so if you prefer, that's the same thing as a vector from the origin to the moving point. So, maybe our point is here, P. So, this vector here -- This vector here is vector OP. And, that's also the position vector r of t. So, just to give you, again, that example -- -- if I take the cycloid for a wheel of radius 1,

and let's say that we are going at unit speed so that the angle that we used as a parameter of time is the same thing as time when the position vector, in this case, we found to be, just to make sure that they have it right, <t - sin(t),1-cos(t)>. OK, that's a formula that you should have in your notes from last time, except we had theta instead of t because we were using the angle. But now I'm saying, we are moving at unit speed, so time and angle are the same thing.

So, now, what's interesting about this is we can analyze the motion in more detail. OK, so, now that we know the position of the point as a function of time, we can try to study how it varies in particular things like the speed and acceleration. OK, so let's start with speed. Well, in fact we can do better than speed. Let's not start with speed. So, speed is a number. It tells you how fast you are going along your trajectory.

I mean, if you're driving in a car, then it tells you how fast you are going. But, unless you have one of these fancy cars with a GPS, it doesn't tell you which direction you're going. And, that's useful information, too, if you're trying to figure out what your trajectory is. So, in fact, there's two aspects to it. One is how fast you are going, and the other is in what direction you're going.

That means actually we should use a vector maybe to think about this. And so, that's called the velocity vector. And, the way we can get it, so, it's called usually V, so, V here stands for velocity more than for vector. And, you just get it by taking the derivative of a position vector with respect to time. Now, it's our first time writing this kind of thing with a vector. So, the basic rule is you can take the derivative of a vector quantity just by taking the derivatives of each component.

OK, so that's just dx/dt, dy/dt, and if you have z component, dz/dt. So, let me -- OK, so -- OK, so let's see what it is for the cycloid. So, an example of a cycloid, well, so what do we get when we take the derivatives of this formula there? Well, so, the derivative of t is 1- cos(t). The derivative of 1 is 0. The derivative of -cos(t) is sin(t). Very good. OK, that's at least one thing you should remember from single variable calculus.

Hopefully you remember even more than that. OK, so that's the velocity vector. It tells us at any time how fast we are going, and in what direction. So, for example, observe. Remember last time at the end of class we were trying to figure out what exactly happens near the bottom point, when we have this motion that seems to stop and go backwards. And, we answered that one way. But, let's try to understand it in terms of velocity.

What if I plug t equals 0 in here? Then, 1- cos(t) is 0, sin(t) is 0. The velocity is 0. So, at the time,at that particular time, our point is actually not moving. Of course, it's been moving just before, and it starts moving just afterwards. It's just the instant, at that particular instant, the speed is zero. So, that's especially maybe a counterintuitive thing, but something is moving. And at that time, it's actually stopped.

Now, let's see, so that's the vector. And, it's useful. But, if you want just the usual speed as a number, then, what will you do? Well, you will just take exactly the magnitude of this vector. So, speed, which is the scalar quantity is going to be just the magnitude of the vector, V. OK, so, in this case, while it would be square root of (1- cost)^2 sin^2(t), and if you expand that, you will get,

let me take a bit more space, it's going to be square root of 1 - 2cos(t) cos^2(t) sin^2(t). It seems to simplify a little bit because we have cos^2 plus sin^2. That's 1. So, it's going to be the square root of 2 - 2cos(t). So, at this point, if I was going to ask you, when is the speed the smallest or the largest? You could answer based on that. See, at t equals 0, well, that turns out to be zero.

The point is not moving. At t equals pi, that ends up being the square root of 2 plus 2, which is 4. So, that's 2. And, that's when you're truly at the top of the arch, and that's when the point is moving the fastest. In fact, they are spending twice as fast as the wheel because the wheel is moving to the right at unit speed, and the wheel is also rotating. So, it's moving to the right and unit speed relative to the center so that the two effects add up, and give you a speed of 2.

Anyway, that's a formula we can get. OK, now, what about acceleration? So, here I should warn you that there is a serious discrepancy between the usual intuitive notion of acceleration, the one that you are aware of when you drive a car and the one that we will be using. So, you might think acceleration is just the directive of speed. If my car goes 55 miles an hour on the highway and it's going a constant speed, it's not accelerating.

But, let's say that I'm taking a really tight turn. Then, I'm going to feel something. There is some force being exerted. And, in fact, there is a sideways acceleration at that point even though the speed is not changing. So, the definition will take effect.

The acceleration is, as a vector, and the acceleration vector is just the derivative of a velocity vector. So, even if the speed is constant, that means, even if a length of the velocity vector stays the same, the velocity vector can still rotate.

And, as it rotates, it uses acceleration. OK, and so this is the notion of acceleration that's relevant to physics when you find F=ma; that's the (a) that you have in mind here. It's a vector. Of course, if you are moving in a straight line, then the two notions are the same. I mean, acceleration is also going to be along the line, and it's going to has to do with the derivative of speed. But, in general, that's not quite the same.

So, for example, let's look at the cycloid. If we take the example of the cycloid, well, what's the derivative of one minus cos(t)? It's sin(t). And, what's the derivative of sin(t)? cos(t), OK. So, the acceleration vector is <sin(t), cos(t)>. So, in particular, let's look at what happens at time t equals zero when the point is not moving. Well, the acceleration vector there will be zero from one.

So, what that means is that if I look at my trajectory at this point, that the acceleration vector is pointing in that direction. It's the unit vector in the vertical direction. So, my point is not moving at that particular time. But, it's accelerating up. So, that means that actually as it comes down, first it's slowing down. Then it stops here, and then it reverses going back up. OK, so that's another way to understand what we were saying last time that the trajectory at that point has a vertical tendency because that's the direction in which the motion is going to occur just before and just after time zero.

OK, any questions about that? No. OK, so I should insist maybe on one thing, which is that, so, we can differentiate vectors just component by component, OK, and we can differentiate vector expressions according to certain rules that we'll see in a moment. One thing that we cannot do, it's not true that the length of dr dt, which is the speed, is equal to the length of dt. OK, this is completely false.

And, they are really not the same. So, if you have to differentiate the length of a vector, but basically you are in trouble. If you really, really want to do it, well, the length of the vector is the square root of the sums of the squares of the components, and from that you can use the formula for the derivative of the square root, and the chain rule, and various other things. And, you can get there.

But, it will not be a very nice expression. There is no simple formula for this kind of thing. Fortunately, we almost never have to compute this kind of thing because, after all, it's not a very relevant quantity. What's more relevant might be this one. This is actually the speed. This one, I don't know what it means. OK. So, let's continue our exploration. So, the next concept that I want to define is that of arc length.

So, arc length is just the distance that you have traveled along the curve, OK? So, if you are in a car, you know, it has mileage counter that tells you how far you've gone, how much fuel you've used if it's a fancy car. And, what it does is it actually integrates the speed of the time to give you the arc length along the trajectory of the car. So, the usual notation that we will have is (s) for arc length.

I'm not quite sure how you get an (s) out of this, but it's the usual notation. OK, so, (s) is for distance traveled along the trajectory. And, so that makes sense, of course, we need to fix a reference point. Maybe on the cycloid, we'd say it's a distance starting on the origin. In general, maybe you would say you start at time, t equals zero. But, it's a convention. If you knew in advance, you could have,

actually, your car's mileage counter to count backwards from the point where the car will die and start walking. I mean, that would be sneaky-freaky, but you could have a negative arc length that gets closer and closer to zero, and gets to zero at the end of a trajectory, or anything you want. I mean, arc length could be positive or negative. Typically it's negative what you are before the reference point, and positive afterwards.

So, now, how does it relate to the things we've seen there? Well, so in particular, how do you relate arc length and time? Well, so, there's a simple relation, which is that the rate of change of arc length versus time, well, that's going to be the speed at which you are moving, OK, because the speed as a scalar quantity tells you how much distance you're covering per unit time. OK, and in fact, to be completely honest,

I should put an absolute value here because there is examples of curves maybe where your motion is going back and forth along the same curve. And then, you don't want to keep counting arc length all the time. Actually, maybe you want to say that the arc length increases and then decreases along the curve. I mean, you get to choose how you count it. But, in this case, if you are moving back and forth, it would make more sense to have the arc length first increase,

then decrease, increase again, and so on. So -- So if you want to know really what the arc length is, then basically the only way to do it is to integrate speed versus time. So, if you wanted to know how long an arch of cycloid is, you have this nice-looking curve; how long is it? Well, you'd have to basically integrate this quantity from t equals zero to 2 pi. And, to say the truth, I don't really know how to integrate that.

So, we don't actually have a formula for the length at this point. However, we'll see one later using a cool trick, and multi-variable calculus. So, for now, we'll just leave the formula like that, and we don't know how long it is. Well, you can put that into your calculator and get the numerical value. But, that's the best I can offer. Now, another useful notion is the unit vector to the trajectory.

So, the usual notation is T hat. It has a hat because it's a unit vector, and T because it's tangent. Now, how do we get this unit vector? So, maybe I should have pointed out before that if you're moving along some trajectory, say you're going in that direction, then when you're at this point, the velocity vector is going to be tangential to the trajectory. It tells you the direction of motion in particular.

So, if you want a unit vector that goes in the same direction, all you have to do is rescale it, so, at its length becomes one. So, it's v divided by a magnitude of v. So, it seems like now we have a lot of different things that should be related in some way. So, let's see what we can say. Well, we can say that dr by dt, so, that's the velocity vector, that's the same thing as if I use the chain rule dr/ds times ds/dt.

OK, so, let's think about this things. So, this guy here we've just seen. That's the same as the speed, OK? So, this one here should be v divided by its length. So, that means this actually should be the unit vector. OK, so, let me rewrite that. It's T ds/dt. So, maybe if I actually stated directly that way, see, I'm just saying the velocity vector has a length and a direction. The length is the speed.

The direction is tangent to the trajectory. So, the speed is ds/dt, and the vector is T hat. And, that's how we get this. So, let's try just to see why dr/ds should be T. Well, let's think of dr/ds. dr/ds means position vector r means you have the origin, which is somewhere out there, and the vector r is here. So, dr/ds means we move by a small amount, delta s along the trajectory a certain distance delta s.

And, we look at how the position vector changes. Well, we'll have a small change. Let me call that vector delta r corresponding to the size, corresponding to the length delta s. And now, delta r should be essentially roughly equal to, well, its direction will be tangent to the trajectory. If I take a small enough interval, then the direction will be almost tensioned to the trajectory times the length of it will be delta s,

the distance that I have traveled. OK, sorry, maybe I should explain that on a separate board. OK, so, let's say that we have that amount of time, delta t. So, let's zoom into that curve. So, we have r at time t. We have r at time t plus delta t. This vector here I will call delta r. The length of this vector is delta s. And, the direction is essentially that of a vector. OK, so, delta s over delta t, that's the distance traveled divided by the time.

That's going to be close to the speed. And, delta r is approximately T times delta s. So, now if I divide both sides by delta t, I get this. And, if I take the limit as delta t turns to zero, then I get the same formula with the derivatives and with an equality. It's an approximation. The approximation becomes better and better if I go to smaller intervals. OK, are there any questions about this?

Yes? Yes, that's correct. OK, so let's be more careful, actually. So, you're asking about whether the delta r is actually strictly tangent to the curve. Is that -- That's correct. Actually, delta r is not strictly tangent to anything. So, maybe I should draw another picture. If I'm going from here to here, then delta r is going to be this arc inside the curve while the vector will be going in this direction, OK?

So, they are not strictly parallel to each other. That's why it's only approximately equal. Similarly, this distance, the length of delta r is not exactly the length along the curve. It's actually a bit shorter. But, if we imagine a smaller and smaller portion of the curve, then this effect of the curve being a curve and not a straight line becomes more and more negligible. If you zoom into the curve sufficiently, then it looks more and more like a straight line.

And then, what I said becomes true in the limit. OK? Any other questions? No? OK. So, what happens next? OK, so let me show you a nice example of why we might want to use vectors to study parametric curves because, after all, a lot of what's here you can just do in coordinates. And, we don't really need vectors. Well, and truly, vectors being a language, you never strictly need it, but it's useful to have a notion of vectors.

So, I want to tell you a bit about Kepler's second law of celestial mechanics. So, that goes back to 1609. So, that's not exactly recent news, OK? But, still I think it's a very interesting example of why you might want to use vector methods to analyze motions. So, what happened back then was Kepler was trying to observe the motion of planets in the sky, and trying to come up with general explanations of how they move.

Before him, people were saying, well, they cannot move in a circle. But maybe it's more complicated than that. We need to add smaller circular motions on top of each other, and so on. They have more and more complicated theories. And then Kepler came with these laws that said basically that planets move in an ellipse around the sun, and that they move in a very specific way along that ellipse.

So, there's actually three laws, but let me just tell you about the second one that has a very nice vector interpretation. So, what Kepler's second law says is that the motion of planets is, first of all, they move in a plane. And second, the area swept out by the line from the sun to the planet is swept at constant time. Sorry, is swept at constant rate. From the sun to the planet, it is swept out by the line at a constant rate.

OK, so that's an interesting law because it tells you, once you know what the orbit of the planet looks like, it tells you how fast it's going to move on that orbit. OK, so let me explain again. So, this law says maybe the sun, let's put the sun here at the origin, and let's have a planet. Well, the planet orbits around the sun -- -- in some trajectory. So, this is supposed to be light blue. Can you see that it's different from white? No? OK, me neither.

[LAUGHTER] OK, it doesn't really matter. So, the planet moves on its orbit. And, if you wait for a certain time, then a bit later it would be here, and then here, and so on. Then, you can look at the amount of area inside this triangular wedge. And, the claim is that the amount of area in here is proportional to the time elapsed. So, in particular, if a planet is closer to the sun, then it has to go faster.

And, if it's farther away from the sun, then it has to go slower so that the area remains proportional to time. So, it's a very sophisticated prediction. And, I think the way he came to it was really just by using a lot of observations, and trying to measure what was true that wasn't true. But, let's try to see how we can understand that in terms of all we know today about mechanics. So, in fact, what happens is that Newton, so Newton was quite a bit later.

That was the late 17th century instead of the beginning of the 17th century. So, he was able to explain this using his laws for gravitational attraction. And, you'll see that if we reformulate Kepler's Law in terms of vectors, and if we work a bit with these vectors, we are going to end up with something that's actually completely obvious to us now. At the time, it was very far from obvious, but to us now to completely obvious.

So, let's try to see, what does Kepler's law say in terms of vectors? OK, so, let's think of what kinds of vectors we might want to have in here. Well, it might be good to think of, maybe, the position vector, and maybe its variation. So, if we wait a certain amount of time, we'll have a vector, delta r, which is the change in position vector a various interval of time. OK, so let's start with the first step.

What's the most complicated thing in here? It's this area swept out by the line. How do we express that area in terms of vectors? Well, I've almost given the answer by drawing this picture, right? If I take a sufficiently small amount of time, this shaded part looks like a triangle. So, we have to find the area of the triangle. Well, we know how to do that now. So, the area is approximately equal to one half of the area of a parallelogram that I could form from these vectors.

And, the area of a parallelogram is given by the magnitude of a cross product. OK, so, I should say, this is the area swept in time delta t. You should think of delta t as relatively small. I mean, the scale of a planet that might still be a few days, but small compared to the other old trajectory. So, let's remember that the amount by which we moved, delta r, is approximately equal to v times delta t,

OK, and just using the definition of a velocity vector. So, let's use that. Sorry, so it's approximately equal to r cross v magnitude times delta t. I can take out the delta t, which is scalar. So, now, what does it mean to say that area is swept at a constant rate? It means this thing is proportional to delta t. So, that means, so, the law says, in fact, that the length of this cross product r cross v equals a constant.

OK, r cross v has constant length. Any questions about that? No? Yes? Yes, let me try to explain that again. So, what I'm claiming is that the length of the cross products r cross v measures the rate at which area is swept by the position vector. I should say, with a vector of one half of this length is the rate at which area is swept. How do we see that? Well, let's take a small time interval, delta t.

In time, delta t, our planet moves by v delta t, OK? So, if it moves by v delta t, it means that this triangle up there has two sides. One is the position vector, r. The other one is v delta t. So, its area is given by one half of the magnitude of a cross product. That's the formula we've seen for the area of a triangle in space. So, the area is one half of the cross product, r, and v delta t, magnitude of the cross product.

So, to say that the rate at which area is swept is constant means that these two are proportional. Area divided by delta t is constant at our time. And so, this is constant. OK, now, what about the other half of the law? Well, it says that the motion is in a plane, and so we have a plane in which the motion takes place. And, it contains, also, the sun. And, it contains the trajectory. So, let's think about that plane.

Well, I claim that the position vector is in the plane. OK, that's what we are saying. But, there is another vector that I know it is in the plane. You could say the position vector at another time, or at any time, but in fact, what's also true is that the velocity vector is in the plane. OK, if I'm moving in the plane, then position and velocity are in there. So, the plane of motion contains r and v.

So, what's the direction of the cross product r cross v? Well, it's the direction that's perpendicular to this plane. So, it's normal to the plane of motion. And, that means, now, that actually we've put the two statements in there into a single form because we are saying r cross v has constant length and constant direction. In fact, in general, maybe I should say something about this. So, if you just look at the position vector, and the velocity vector for any motion at any given time, then together, they determine some plane.

And, that's the plane that contains the origin, the point, and the velocity vector. If you want, it's the plane in which the motion seems to be going at the given time. Now, of course, if your motion is not in a plane, then that plane will change. It's, however, instant, if a plane in which the motion is taking place at a given time. And, to say that the motion actually stays in that plane forever means that this guy will not change direction.

OK, so -- [LAUGHTER] [APPLAUSE] OK, so, Kepler's second law is actually equivalent to saying that r cross v equals a constant vector, OK? That's what the law says. So, in terms of derivatives, it means d by dt of r cross v is the zero vector. OK, now, so there's an interesting thing to note, which is that we can use the usual product rule for derivatives with vector expressions, with dot products or cross products.

There's only one catch, which is that when we differentiate a cross product, we have to be careful that the guy on the left stays on the left. The guy on the right stays on the right. OK, so, if you know that uv prime equals u prime v plus uv prime, then you are safe. If you know it as u prime v cross v prime u, then you are not safe. OK, so it's the only thing to watch for. So, product rule is OK for taking the derivative of a dot product.

There, you don't actually even need to be very careful about all the things or the derivative of a cross product. There you just need to be a little bit more careful. OK, so, now that we know that, we can write this as dr/dt cross v plus r cross dv/dt, OK? Well, let's reformulate things slightly. So, dr dt already has a name. In fact, that's v. OK, that's what we call the velocity vector. So, this is v cross v plus r cross, what is dv/dt?

That's the acceleration, a, equals zero. OK, so now what's the next step? Well, we know what v cross v is because, remember, a vector cross itself is always zero, OK? So, this is the same r cross a equals zero, and that's the same as saying that the cross product of two vectors is zero exactly when the parallelogram of the form has no area. And, the way in which that happens is if they are actually parallel to each other.

So, that means the acceleration is parallel to the position. OK, so, in fact, what Kepler's second law says is that the acceleration is parallel to the position vector. And, since we know that acceleration is caused by a force that's equivalent to the fact that the gravitational force -- -- is parallel to the position vector, that means, well, if you have the sun here at the origin, and if you have your planets, well, the gravitational force caused by the sun should go along this line.

In fact, the law doesn't even say whether it's going towards the sun or away from the sun. Well, what we know now is that, of course, the attraction is towards the sun. But, Kepler's law would also be true, actually, if things were going away. So, in particular, say, electric force also has this property of being towards the central charge. So, actually, if you look at motion of charged particles in an electric field caused by a point charged particle, it also satisfies Kepler's law, satisfies the same law.

OK, that's the end for today, thanks.