18.03 Class 16, March 15, 2006

Frequency response

[1] Frequency response: without damping

First recall the Harmonic Oscillator: x" + omega_n^2 x = 0 :

The spring constant is k = omega_n^2 .

Solutions are arbitrary sinusoids with circular frequency omega_n , the "natural frequency" of the system.

Drive it sinusoidally: x" + omega_n^2 x = omega_n^2 A cos(omega t)

I am driving the system through the spring, with a plunger moving sinusoidally with amplitude 1 . cos(omega t) is the "physical signal," as opposed to the force, or "complete signal" omega_n^2 cos(omega t) . We regard the plunger position as the system input.

We solved this by luckily trying x_p = B cos(omega t) and solving for B .
Let's do it using ERF:

z" + omega_n^2 z = omega_n^2 A e^{i omega t}

z_p = A (omega_n^2 / (omega_n^2 - omega ^2)) e^{i omega t}

No damping ===> denominator is real and so

x_p = A (omega_n^2 / (omega_n^2 - omega^2)) cos(omega t)

This is ok unless omega = omega_n , in which case the system is in resonance with the signal.

[2] Bode Plots.

The "gain" is the ratio of the output amplitude to the physical signal amplitude. In this case,

gain(omega) = |omega_n^2/(omega_n^2 - omega^2)|

This has graph which starts with gain(0) = 1 , increases to infinity when omega = omega_n, and then falls towards zero when omega > omega_n .

There's a phase transition too: if we write the solution as

x_p = gain . cos(omega t - phi)

then phi = 0 for omega < omega_n

and phi = - pi for omega > omega_n .

The traditional thing to graph is - phi rather than phi: this graph is constant zero for omega < omega_n and then switches discontinuously to - phi = -pi for omega > omega_n .

[3] Damped systems: Frequency response

Drive this system sinusoidally, through the spring:

$$x'' + bx' + kx = k A \cos(\omega t)$$

We continue to write $k = \omega_n^2$ and call ω_n the "natural circular frequency" of the system.

Physical input $\cos(\omega t)$ has amplitude A. The gain is the amplitude
of the sinusoidal output divided by A.

I displayed "Amplitude and Phase, Second Order" and set $k = 3.24$ and $b = .5$. In it, $B = 1$.

ERF: $z'' + bz' + kz = k B e^{rt}$

$$z_p = W(r) B e^{rt} , \quad W(r) = k / p(r) .$$

$W(r)$ is the "transfer function." Sinusoidal input means $r = i \omega$:

$$W(i \omega) = \omega_n^2 / ((\omega_n^2 - \omega^2) + i b \omega)$$

This is the "complex gain."

In the expression

$$x_p = \text{gain} \cdot \cos(\omega t - \phi)$$

I claim that

$$- \phi = \mathrm{Arg}(W(i \omega))$$

$$\text{gain} = |W(i \omega)|$$

Proof: $W(i \omega) = |W(i \omega)| e^{- i \phi}$

Then

$$z_p = |W(i \omega)| e^{- i \phi} e^{i \omega t}$$

$$= |W(i \omega)| e^{i(\omega t - \phi)}$$

which has real part

$$x_p = |W(i \omega)| \cos(\omega t - \phi)$$

Compare this with the undamped case: there is an imaginary part to the denominator. This causes two effects:

(1) The magnitude of the denominator is increased, causing the gain to decrease. Especially: the denominator can never be zero anymore, no matter what ω is, since it has a nonzero imaginary part. Thus you never encounter true resonance with a sinusoidal signal, if there is any damping.

(2) A phase lag appears. Since $b > 0$, the imaginary part of the denominator is positive, so

$$\mathrm{Im}\, W(i\omega) < 0 \qquad \text{unless} \quad \omega = 0$$

which says that $0 < \phi < \pi$.

[5] More explicitly,

$$W(i\omega) = k / ((k - \omega^2) + i b \omega)$$

When $\omega = 0$ this is 1 : gain 1, phase lag 0.

As omega increases, $W(i\omega)$ sweeps out a curve in the complex plane.

Gain:

$$|W(i\omega)| = k / \sqrt{(k - \omega^2)^2 + b^2 \omega^2}$$

If b is small, the gain is large (though not infinite) when ω is near to the natural frequency of the system, since the first term in the denominator is small. This is NEAR RESONANCE.

When omega gets very large, the denominator is roughly ω^2, so the gain tails off like k/ω^2.

As b grows larger, the second term dominates and for modest values of omega

$$|W(i\omega)| \sim k / (b \omega)$$

This doesn't have maxima anymore; for large b there is no near-resonant peak.
Eg in the tool, when $k = 1$ the resonant peak vanishes when $b = \sqrt{2}$.

Phase lag: Since $k = \omega_n^2$, $W(i\omega)$ is purely imaginary when $\omega = \omega_n$: that's when the phase lag of the solution is 90 degrees.

$$|W(i\omega_n)| = \omega_n / b$$

When $b = .5$ and $k = 3.24$, $\omega_n = 1.8$ and indeed when $\omega = 1.8$,
the phase lag is exactly 90 degrees and the gain is $1.8/.5 = 3.60$.

The gain won't be maximal then (think of the case of b large), but you should expect it to be relatively large.