Recitation 14, April 4, 2006

Fourier Series: Playing around

Solution suggestions

$$g(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$

$$+ b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$

The period of the function $g(x)$ is $2L$. The Fourier coefficients are defined as the numbers fitting into this expression. They can be calculated using the integral formulas

$$a_n = \frac{1}{L}\int_{-L}^{L} g(x) \cos\left(\frac{n\pi x}{L}\right)\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} g(x) \sin\left(\frac{n\pi x}{L}\right)\,dx$$

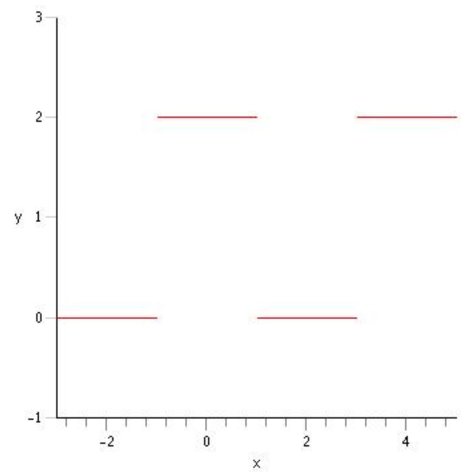
but often they can be found more easily than this, starting from some basic examples. One basic example: $\mathrm{sq}(x)$ is the odd function of period 2π such that $\mathrm{sq}(t) = 1$ for $0 < t < \pi$.

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right) = \frac{4}{\pi}\sum_{k \text{ odd}} \frac{\sin(kt)}{k}$$

[I have not used the summation convention in lecture; it adds to the confusion. If you discuss it please do so carefully.]

1. Find the Fourier series for $g(x)$, periodic of period 4, such that $g(x) = 2$ for $-1 < x < 1$ and $g(x) = 0$ for $1 < x < 3$ using the integral formulas. First step: is it even or odd or neither? If it is one of these, note that the interval of integration can be halved. Do the integral. Tabulate the values of the anti-derivative at one end point against the number n. Finally write out the sum.

Ans. We see that the function is even as it is symmetric around $x = 0$, or $g(-x) = g(x)$.



Now $\sin(kt)$ is an odd function, the product of an odd and an even function is odd, thus $g(x)\sin(n\pi x)$ is odd. If we integrate an odd function from $-L$ to L we will always get zero. Thus, we have $b_n = 0$ for all n. Similarly, the product of an even and another even function is even again. Thus,

$$a_n = \frac{2}{L}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)\,dx\ .$$

Now, $2L$ equals the period which is 4, so $L = 2$. Plugging in the definition of $g(x)$ – remember it's zero between 1 and 2 – we have

$$a_n = \int_0^1 \cos\left(\frac{n\pi x}{2}\right)\,dx\ .$$

Let's start with a_0, it's simplify

$$a_0 = \int_0^1 2dx = 2\ .$$

Now, let's compute the other a_n's:

$$\begin{aligned} a_n &= \int_0^1 2\cos\left(\frac{n\pi x}{2}\right)\,dx \\ &= \frac{4}{n\pi}\int_0^{\frac{n\pi}{2}} \cos t\,dt \\ &= \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\ . \end{aligned}$$

We know that $\sin(k\pi) = 0$ if k is an integer. Therefore, a_n must be zero if n is even. We also remember that $\sin(n\pi/2) = 1$ for $n = 1, 5, 9, 13, \ldots$ and $\sin(n\pi/2) = -1$ for $n = 3, 7, 11, \ldots$. In conclusion, we have computed the following coefficients

a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	$\ldots$
2	$\frac{4}{\pi}$	0	$-\frac{4}{3\pi}$	0	$\frac{4}{5\pi}$	0	$-\frac{4}{7\pi}$	$\ldots$

and all the b_n's are zero. So the first terms in the Fourier series look like

$$g(x) = 1 + \frac{4}{\pi}\Big\{ \cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\cos\left(\frac{5\pi x}{2}\right) + \dots \Big\} .$$

If we want to write it with the summation convention we would write

$$g(x) = 2 + \frac{4}{\pi}\sum_{n \text{ odd}} (-1)^{\frac{n-1}{2}} \frac{\cos\left(\frac{n\pi x}{2}\right)}{n}$$

2. Now write the same function in terms of $\mathrm{sq}(t)$ by suitable change of variables, shifting, and scaling, and then use the Fourier series for $\mathrm{sq}(t)$ to obtain the Fourier series for $g(x)$.

Ans. First we notice that

$$g(x-1) - 1 = \begin{cases} 1 & 0 < x < 2 \\ -1 & -2 < x < 0 \end{cases}$$

Thus, $g(x-1) - 1$ is odd and has period 4. We want to express it in terms of $\mathrm{sq}(t)$ which is

$$\mathrm{sq}(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & -\pi < t < 0 \end{cases}$$

Therefore, if we set $t = \frac{2\pi}{4}x = \frac{\pi}{2}x$ the two functions are the same, i.e.

$$g(x-1) - 1 = \mathrm{sq}(\frac{\pi}{2}x) .$$

Since we want to express $g(x)$ through $\mathrm{sq}(t)$ we take $x' = x - 1$ or $x = x' + 1$. Then,

$$g(x') = 1 + \mathrm{sq}\left(\frac{\pi}{2}x' + \frac{\pi}{2}\right)$$

or if we drop the prime

$$g(x) = 1 + \mathrm{sq}\left(\frac{\pi}{2}x + \frac{\pi}{2}\right)$$

Therefore, we have

$$g(x) = 1 + \frac{4}{\pi}\sum_{k \text{ odd}} \frac{\sin\left(\frac{k\pi}{2}(x+1)\right)}{k} .$$

We notice that $\sin(y + \frac{k\pi}{2})$ is $\cos(y)$ for $k = 1, 5, 9, 13, \dots$ and $-\cos(y)$ for $k = 2, 7, 11, \dots$. Therefore, we have

$$g(x) = 1 + \frac{4}{\pi}\sum_{k \text{ odd}} (-1)^{\frac{k-1}{2}} \frac{\cos\left(\frac{k\pi}{2}x\right)}{k} ,$$

which agrees with our answer from **(1)**.

3. What is the Fourier series for $\sin^2 t$?

Ans. The function $\sin^2 t$ is even and of period π, $L = \pi/2$. Therefore, all coefficients b_n must be zero. Instead of computing the coefficients a_n let us do the following: By Euler's identity we have

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) .$$

and thus

$$\sin^2 x = -\frac{1}{4}\left(e^{2ix} - 2 + e^{-2ix}\right) = \frac{1}{2}\left(1 - \frac{e^{2ix} + e^{-2ix}}{2}\right) .$$

That is the well-known trigonometric identity

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x)) = \frac{1}{2} - \frac{1}{2}\cos(2x) .$$

By comparison we see that $a_0 = 1$ and $a_1 = -\frac{1}{2}$ and all other a_n are zero.

4. Explain why any function $g(x)$ is a sum of an even function and an odd function in just one way. What is the even part of e^x? What is the odd part?

Ans. We can write any function $f(x)$ as

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd}} .$$

Replacing x by $x' = -x$ we see that the first part is even since

$$\frac{f(x') + f(-x')}{2} = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} .$$

Similarly, we can compute for the second part

$$\frac{f(x') - f(-x')}{2} = \frac{f(-x) - f(x)}{2} = -\frac{f(x) + f(-x)}{2} .$$

Therefore, the second part is odd. Accordingly, for $f(x) = e^x$ the even part is

$$\frac{f(x) + f(-x)}{2} = \frac{e^x + e^{-x}}{2} = \cosh(x) ,$$

which is in fact an even function, i.e. $\cosh(-x) = \cosh(x)$. Similarly, we obtain for the second part

$$\frac{f(x) - f(-x)}{2} = \frac{e^x - e^{-x}}{2} = \sinh(x) ,$$

which is an odd function, i.e. $\sinh(-x) = -\sinh(x)$.