Recitation 14, April 4, 2006

Fourier Series: Playing around

$$g(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$

$$+ b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$

The period of $g(x)$ is $2L$. The Fourier coefficients are defined as the numbers fitting into this expression. They can be calculated using the integral formulas

$$a_n = \frac{1}{L} \int_{-L}^{L} g(x) \cos\left(\frac{n\pi x}{L}\right)\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} g(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$$

but often they can be found more easily than this, starting from some basic examples. One basic example: $\mathrm{sq}(x)$ is the odd function of period 2π such that $\mathrm{sq}(t) = 1$ for $0 < t < \pi$.

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right) = \frac{4}{\pi} \sum_{k \text{ odd}} \frac{\sin(kt)}{k}$$

1. Find the Fourier series for $g(x)$, periodic of period 4, such that $g(x) = 2$ for $-1 < x < 1$ and $g(x) = 0$ for $1 < x < 3$ using the integral formulas. First step: is it even or odd or neither? If it is one of these, note that the interval of integration can be halved. Do the integral. Tabulate the values of the anti-derivative at one end point against the number n. Finally write out the sum.

2. Now write the same function in terms of $\mathrm{sq}(t)$ by suitable change of variables, shifting, and scaling, and then use the Fourier series for $\mathrm{sq}(t)$ to obtain the Fourier series for $g(x)$.

3. What is the Fourier series for $\sin^2 t$?

4. Explain why any function $g(x)$ is a sum of an even function and an odd function in just one way. What is the even part of e^x? What is the odd part?