18.03 Class 15, March 13, 2004

Operators: Exponential shift law
Undetermined coefficients

[1] Operators. The ERF is based on the following calculation:

$$D e^{rt} = r e^{rt} = rI e^{rt}$$

so $D^n e^{rt} = r^n I e^{rt}$

and $(a_n D^n + ... + a_0 I) e^{rt} = (a_n r^n + ... + a_0) e^{rt}$

or $p(D) e^{rt} = p(r) e^{rt}$

So to solve $p(D) x = A e^{rt}$, try $x_p = B e^{rt}$;

$$p(D) (B e^{rt}) = B p(D) e^{rt} = B p(r) e^{rt}$$

so we should take $B = A/p(r)$: $x_p = e^{rt}/p(r)$.

What if $p(r) = 0$? eg $x'' - x = e^{-t}$. (*)

The key to solving this problem is the behavior of D on products:

$$(d/dt) (xy) = x' y + x y'$$

In terms of operators:

$$D(vu) = v Du + u Dv$$

Especially:

$$\begin{aligned} D(e^{rt} u) &= e^{rt} Du + u r e^{rt} \\ &= e^{rt} (Du + ru) \\ &= e^{rt} (D + rI) u \end{aligned}$$

Apply D again:

$$\begin{aligned} D^2 (e^{rt} u) &= D(e^{rt} (D+rI)u) \\ &= e^{rt} (D+rI)(D+rI) u \\ &= e^{rt} (D+rI)^2 u \end{aligned}$$

Use: let's try a variation of parameters approach to solving (*):

Try for $x = e^{-t} u$

Then $D^2 x = e^{-t} (D-I)^2 u$

$$\begin{array}{rrcl} & D^2 x & = & e^{-t} (D-I)^2 u \\ -1] & x & = & e^{-t} I u \\ \hline & e^{-t} & = & e^{-t} ((D-I)^2 - I) u \end{array}$$

so want $((D-I)^2 - I) u = 1$

or $(D^2 - 2D) u = 1$ i.e. $u'' - 2u' = 1$

and this we can do by "reduction of order": say $v = u'$, so we have $v' - 2v = 1$. With a constant right hand side, you get a constant solution (unless the coefficient of v is zero): $v = -1/2$.

Then $u = -t/2$ and $x_p = - t e^{-t}/2$.

Putting this together we get the "Exponential Shift Law":

$$p(D) (e^{rt} u) = e^{rt} p(D+rI) u$$

and using it we find:

ERF2: If $p(r) = 0$ then a solution of $p(D) x = A e^{rt}$ is given by

$$x_p = (a/p'(r)) t e^{rt} \qquad \text{provided } p'(r) \text{ is not zero.}$$

In our case, $p(s) = s^2 - 1$ so $p'(s) = 2s$ and $p'(-1) = -2$, and you recover the solution we worked out.

This is described in more detail in the Notes.

[2] Polynomial signals: Undetermined coefficients.

Notice that if $p(s) = a_n s^n + a_{(n-1)} s^{n-1} + \ldots + a_1 s + a_0$ then $p(0) = a_0$.

Theorem (Undetermined coefficients) Take $q(t) = b_k t^k + \ldots + b_1 t + b_0$.
$p(D)x = q(t)$ has exactly one solution which is polynomial of degree less than or equal to k, provided that $p(0) = a_0$ is not zero.

Proof by example:

$$x'' + 2x' + 3x = t^2 + 1$$

The theorem applies since 3 is not 0 : there is a solution of the form

$$x = at^2 + bt + c$$

To find a , b , c , plug in:

$$\begin{array}{llrrr}
3) & x &= at^2 & + bt & + c \\
2) & x' &= & 2at & + b \\
1) & x'' &= & & 2a \\
\hline
\end{array}$$

$$t^2 + 1 = 3at^2 + (3b+4a)t + (3c+2b+4a)$$

The coefficients must be equal. Since 3 is not zero, we can divide by it to find $a = 1/3$. Then $b = -(1/3)4a = -4/9$. Finally, $c = (1/3)(1-2b-4a) = 11/27$. So

$$xp = (1/3)t^2 - (4/9)t + (11/27)$$

If $a_0 = 0$ we can use "reduction of order":

$$x'' + x' = t$$

Substitute $u = x'$ so $u' + u = t$

$$\begin{aligned} u &= at + b \\ u' &= \phantom{at + {}} a \\ \hline t &= at + (a+b) \end{aligned}$$

$a = 1$, $b = -1$, $u = t - 1$ [check it!] , $x = t^2/2 - t + c$.