Recitation 9, March 9, 2006

Second order linear equations

Solution suggestions

1. Show that $\ddot{x} + 4x = 10$ has a constant solution.

What is the general real solution?

Ans. We want to find the constant colution $x = C$. Since $\dot{x} = 0$, we obtain $4C = 10$ or $C = \frac{5}{2}$. The constant solution is $x_p = \frac{5}{2}$.

To find the general solution, we have to find the solutions of $\ddot{x} + 4x = 0$ first. Plugging in $x = e^{rt}$ we obtain the characteristic equation $r^2 + 4 = 0$. The roots are $r = \pm 2i$. This means that the solutions are the undamped sin and consine. Thus, the solution to $\ddot{x} + 4x = 0$ is $x_h = c_1 \sin(2t) + c_2 \cos(2t)$. Or we could write it as $x_h = A\cos(2t - \phi)$. In both case we see that the solution contains two parameters. Let's combine the two solutions to $x = x_p + x_h$. We have $\dot{x} = \dot{x}_p + \dot{x}_h$, and $\ddot{x} = \ddot{x}_p + \ddot{x}_h$. We obtain

$$\ddot{x} + 4x = \Big(\ddot{x}_p + \ddot{x}_h\Big) + 4\Big(x_p + x_h\Big) = \underbrace{\Big(\ddot{x}_p + 4x_p\Big)}_{=10} + \underbrace{\Big(\ddot{x}_h + 4x_h\Big)}_{=0} = 10 \ .$$

Thus, the general solution for $\ddot{x} + 4x = 10$ is

$$x = \frac{5}{2} + c_1 \sin(2t) + c_2 \cos(2t) \ .$$

2. Find A so that $A \sin t$ is a solution of $\ddot{x} + 4x = 3\sin(t)$.

Ans. With $x = A\sin t$ we have $\dot{x} = A\cos t$ and $\ddot{x} = -A\sin t$. Thus, we obtain

$$\ddot{x} + 4x = 3A \ \sin t \ .$$

As the RHS has to be equal to $3\sin t$, we obtain $A = 1$.

3. What is the general solution of $\ddot{x} + 4x = 3\sin t + 10$?

Ans. In problem 2 we found a solution to $\ddot{x} + 4x = 3\sin(t)$ which was $x_1 = 3\sin t$. In problem 1 we found the general solution to $\ddot{x} + 4x = 10$ which was

$$x_2 = \frac{5}{2} + c_1 \sin(2t) + c_2 \cos(2t) \ .$$

Now, let's form the solution $x = x_1 + x_2$. We compute

$$\ddot{x} + 4x = \Big(\ddot{x}_1 + \ddot{x}_2\Big) + 4\Big(x_1 + x_2\Big) = \underbrace{\Big(\ddot{x}_1 + 4x_1\Big)}_{=3\sin t} + \underbrace{\Big(\ddot{x}_2 + 4x_2\Big)}_{=10} = 3\sin t + 10 \ .$$

Therefore, we have found the general solution

$$x = x_1 + x_2 = 3\sin t + \frac{5}{2} + c_1 \sin(2t) + c_2 \cos(2t) \ .$$

4. What is the solution of $\ddot{x} + 2\dot{x} + 65x = 0$ with $x(0) = 1$ and $\dot{x}(0) = 7$?

Sketch the graph of this function. Begin by sketching the graph of the exponential factor.

At what times does this function peak? (For this it may be easier to use the "rectangular" form of the solutions, using a linear combination of cosine and sine rather than an amplitude and phase shift.)

Ans. Let's find the general solution to $\ddot{x} + 2\dot{x} + 65x = 0$. Plugging in $x = e^{rt}$ we obtain the characteristic equation $r^2 + 2r + 65 = 0$. By completing the square we obtain $(r+1)^2+64 = 0$. Thus, the roots are $r = -1\pm\sqrt{-64} = 2\pm 8i$. Picking the plus sign we obtain $e^{(-1+8i)t}$ which has real part $e^{-t}\cos(8t)$ and imaginary part $e^{-t}\sin(8t)$. The general solution is

$$x = e^{-t}\Big(c_1\cos(8t) + c_2\sin(8t)\Big) .$$

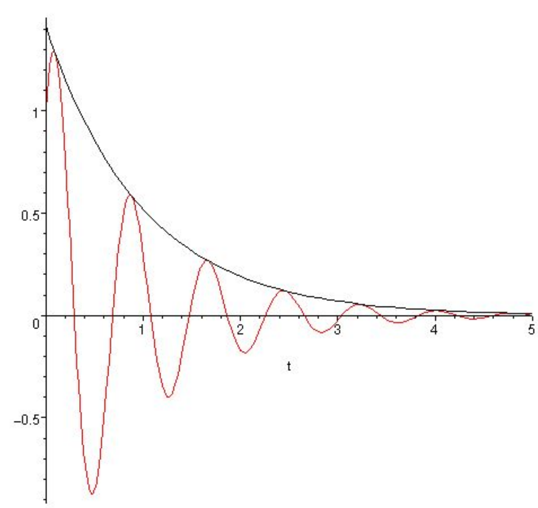
Now, we want to find the solution that satisfies $x(0) = 1$ and $\dot{x}(0) = 0$. We find $x(0) = c_1 = 1$. We compute

$$\dot{x}(t) = -x(t) + e^{-t}\Big(- 8c_1\sin(8t) + 8c_2\cos(8t)\Big) ,$$

and $\dot{x}(0) = -1+8c_2 = 7$, thus $c_2 = 1$. The unique solution of $\ddot{x}+2\dot{x}+65x = 0$ with $x(0) = 1$ and $\dot{x}(0) = 7$ is

$$x(t) = e^{-t}\Big(\cos(8t) + \sin(8t)\Big) = \sqrt{2}e^{-t}\cos(8t - \frac{\pi}{4}) .$$

Here are the graphs of the function $\sqrt{2}\,e^{-t}$ (black) and $e^{-t}\Big(\cos(8t) + \sin(8t)\Big)$ (red)



We see that the zeros of x are at $8t - \frac{\pi}{4} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2} \ldots$. For the critical points we have to find t such that $\dot{x} = 0$. We compute

$$\dot{x} = -\sqrt{2}\,e^{-t}\cos(8t-\frac{\pi}{4}) - \sqrt{2}\,8e^{-t}\sin(8t-\frac{\pi}{4}) = -\sqrt{2}\,e^{-t}\Big(\cos(8t-\frac{\pi}{4}) + 8\sin(8t-\frac{\pi}{4})\Big) .$$

We can write the RHS as one consine function with phase, i.e.

$$\dot{x} = -\sqrt{130}\, e^{-t} \cos(8t - \frac{\pi}{4} - \phi) ,$$

with $\tan\phi = 8$. The RHS is zero and we have a critical point exactly when

$$0 = \cos(8t - \frac{\pi}{4} - \phi) .$$

The *maxima* occur exactly when the cosine changes it sign from negative to positive since there is an overall minus in $\dot{x}$. Thus, we find for the position of the maxima $8t - \frac{\pi}{4} - \phi = \ldots, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2} \ldots$. Due to the damping the position of the maxima is shifted a bit ($\phi \approx 0.9\frac{\pi}{2}$) compared to the position they would have for a cosine function alone without damping ($\phi = \frac{\pi}{2}$).

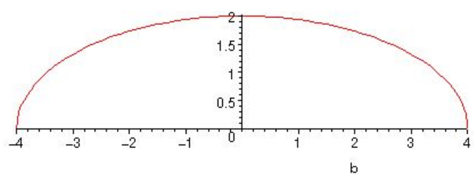
5. For what non-negative values of b does $\ddot{x} + b\dot{x} + 4x = 0$ exhibit solutions which are damped sinusoids?

What is the circular pseudofrequency of these solutions, as a function of the damping constant b? Sketch the graph of this function. Sketch the the graph of the pseudoperiod as a function of b. Can you envision the graphs of solutions of these equations, and how they vary as b varies?

By plugging e^{rt} into $\ddot{x} + b\dot{x} + 4x = 0$ we obtain the characteristic equation $r^2 + br + 4 = 0$. By completing the square we get $(r + \frac{b}{2})^2 + (4 - \frac{b^2}{4}) = 0$. Thus, the roots are complex if $\frac{b^2}{4} < 4$ or $|b| < 4$. Then, we obtain $r = -\frac{b}{2} \pm i\sqrt{4 - \frac{b^2}{4}}$. Thus, the general solution is

$$x(t) = e^{-\frac{b}{2}t}\Big(c_1 \cos(\omega t) + c_2 \sin(\omega t)\Big)$$

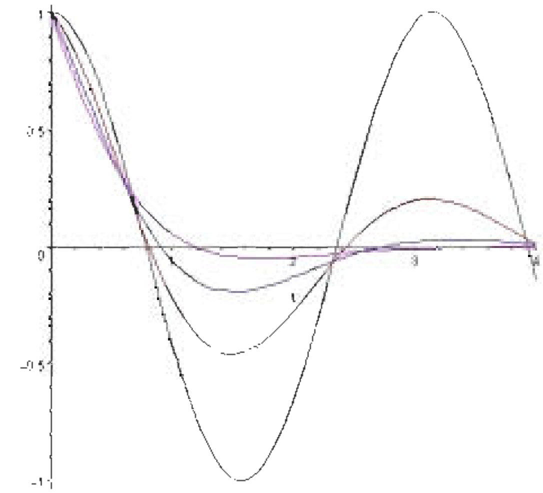
with $\omega = \sqrt{4 - \frac{b^2}{4}}$. If we plot ω as a function of b we obtain



The general solution can be written as

$$x(t) = e^{-\frac{b}{2}t} \cos(\omega t - \phi) .$$

For the following graph, we have picked $\phi = 0$. The graph shows the solution for $b = 0$ (black), $b = 1$ (red), $b = 2$ (magenta), $b = 3$ (blue).


1
0.5
0
-0.5
-1
2
3
4
t