

## G. Graphical and Numerical Methods

In studying the first-order ODE

$$(1) \quad \frac{dy}{dx} = f(x, y),$$

the main emphasis is on learning different ways of finding explicit solutions. But you should realize that most first-order equations cannot be solved explicitly. For such equations, one resorts to graphical and numerical methods. Carried out by hand, the graphical methods give rough qualitative information about how the graphs of solutions to (1) look geometrically. The numerical methods then give the actual graphs to as great an accuracy as desired; the computer does the numerical work, and plots the solutions.

### 1. Graphical methods.

The graphical methods are based on the construction of what is called a **direction field** for the equation (1). To get this, we imagine that through each point  $(x, y)$  of the plane is drawn a little line segment whose slope is  $f(x, y)$ . In practice, the segments are drawn in at a representative set of points in the plane; if the computer draws them, the points are evenly spaced in both directions, forming a lattice. If drawn by hand, however, they are not, because a different procedure is used, better adapted to people.

To construct a direction field by hand, draw in lightly, or in dashed lines, what are called the **isoclines** for the equation (1). These are the one-parameter family of curves given by the equations

$$(2) \quad f(x, y) = c, \quad c \text{ constant.}$$

Along the isocline given by the equation (2), the line segments all have the same slope  $c$ ; this makes it easy to draw in those line segments, and you can put in as many as you want. (Note: "iso-cline" = "equal slope".)

The picture shows a direction field for the equation

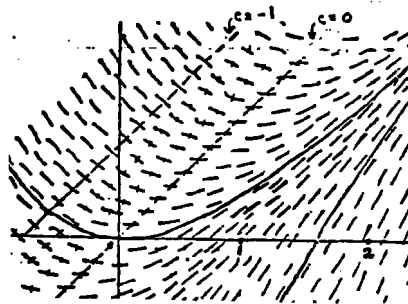
$$y' = x - y.$$

The isoclines are the lines  $x - y = c$ , two of which are shown in dashed lines, corresponding to the values  $c = 0, -1$ . (Use dashed lines for isoclines).

Once you have sketched the direction field for the equation (1) by drawing some isoclines and drawing in little line segments along each of them, the next step is to draw in curves which are at each point tangent to the line segment at that point. Such curves are called **integral curves** or *solution curves* for the direction field. Their significance is this:

$$(3) \quad \text{The integral curves are the graphs of the solutions to } y' = f(x, y).$$

**Proof.** Suppose the integral curve  $C$  is represented near the point  $(x, y)$  by the graph of the function  $y = y(x)$ . To say that  $C$  is an integral curve is the same as saying



$$\text{slope of } C \text{ at } (x, y) = \text{slope of the direction field at } (x, y);$$

from the way the direction field is defined, this is the same as saying

$$y'(x) = f(x, y).$$

But this last equation exactly says that  $y(x)$  is a solution to (1).  $\square$

We may summarize things by saying, the direction field gives a picture of the first-order equation (1), and its integral curves give a picture of the solutions to (1).

Two integral curves (in solid lines) have been drawn for the equation  $y' = x - y$ . In general, by sketching in a few integral curves, one can often get some feeling for the behavior of the solutions. The problems will illustrate. Even when the equation can be solved exactly, sometimes you learn more about the solutions by sketching a direction field and some integral curves, than by putting numerical values into exact solutions and plotting them.

There is a theorem about the integral curves which often helps in sketching them.

#### **Integral Curve Theorem.**

(i) *If  $f(x, y)$  is defined in a region of the  $xy$ -plane, then integral curves of  $y' = f(x, y)$  cannot cross at a positive angle anywhere in that region.*

(ii) *If  $f_y(x, y)$  is continuous in the region, then integral curves cannot even be tangent in that region.*

A convenient summary of both statements is (here "smooth" = continuously differentiable):

#### **Intersection Principle**

(3) *Integral curves of  $y' = f(x, y)$  cannot intersect wherever  $f(x, y)$  is smooth.*

**Proof of the Theorem.** The first statement (i) is easy, for at any point  $(x_0, y_0)$  where they crossed, the two integral curves would have to have the same slope, namely  $f(x_0, y_0)$ . So they cannot cross at a positive angle.

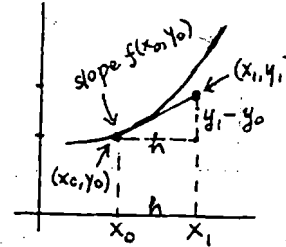
The second statement (ii) is a consequence of the uniqueness theorem for first-order ODE's; it will be taken up then when we study that theorem. Essentially, the hypothesis guarantees that through each point  $(x_0, y_0)$  of the region, there is a unique solution to the ODE, which means there is a unique integral curve through that point. So two integral curves cannot intersect — in particular, they cannot be tangent — at any point where  $f(x, y)$  has continuous derivatives.

## 2. Euler's numerical method.

The graphical method gives you a quick feel for how the integral curves behave. But when they must be known accurately and the equation cannot be solved exactly, numerical methods are used. The simplest method is called **Euler's method**. Here is its geometric description.

We want to calculate the solution (integral curve) to  $y' = f(x, y)$  passing through  $(x_0, y_0)$ . It is shown as a curve in the picture.

We choose a step size  $h$ . Starting at  $(x_0, y_0)$ , over the interval  $[x_0, x_0 + h]$ , we approximate the integral curve by the tangent line: the line having slope  $f(x_0, y_0)$ . (This is the slope of the integral curve, since  $y' = f(x, y)$ .)



This takes us as far as the point  $(x_1, y_1)$ , which is calculated by the equations (see the picture)

$$x_1 = x_0 + h, \quad y_1 = y_0 + h f(x_0, y_0).$$

Now we are at  $(x_1, y_1)$ . We repeat the process, using as the new approximation to the integral curve the line segment having slope  $f(x_1, y_1)$ . This takes us as far as the next point  $(x_2, y_2)$ , where

$$x_2 = x_1 + h, \quad y_2 = y_1 + h f(x_1, y_1).$$

We continue in the same way. The general formulas telling us how to get from the  $(n-1)$ -st point to the  $n$ -th point are

$$(4) \quad x_n = x_{n-1} + h, \quad y_n = y_{n-1} + h f(x_{n-1}, y_{n-1}).$$

In this way, we get an approximation to the integral curve consisting of line segments joining the points  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $\dots$

In doing a few steps of Euler's method by hand, as you are asked to do in some of the exercises to get a feel for the method, it's best to arrange the work systematically in a table.

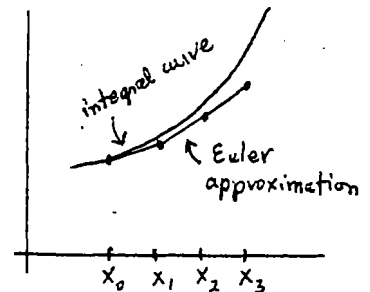
**Example 1.** For the IVP:  $y' = x^2 - y^2$ ,  $y(1) = 0$ , use Euler's method with step size .1 to find  $y(1.2)$ .

**Solution.** We use  $f(x, y) = x^2 - y^2$ ,  $h = .1$ , and (4) above to find  $x_n$  and  $y_n$ :

$n$	$x_n$	$y_n$	$f(x_n, y_n)$	$hf(x_n, y_n)$
0	1	0	1	.1
1	1.1	.1	1.20	.12
2	1.2	.22		

**Remarks.** Euler's method becomes more accurate the smaller the step-size  $h$  is taken. But if  $h$  is too small, round-off errors can appear, particularly on a pocket calculator.

As the picture suggests, the errors in Euler's method will accumulate if the integral curve is convex (concave up) or concave (concave down). Refinements of Euler's method are aimed at using as the slope for the line segment at  $(x_n, y_n)$  a value which will correct for the convexity or concavity, and thus make the next point  $(x_{n+1}, y_{n+1})$  closer to the true integral curve. We will study some of these. The book in Chapter 6 has numerical examples illustrating Euler's method and its refinements.



### Exercises: Section 1C